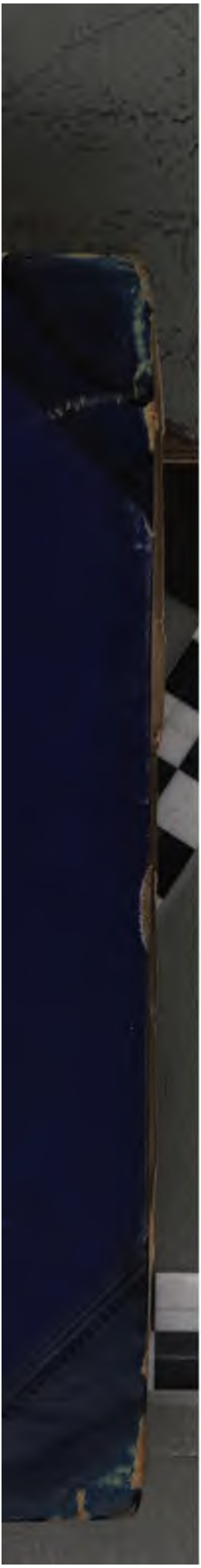

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XXIV.

NOVEMBER, 1892, TO NOVEMBER, 1893.

LONDON:

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1893.

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LONDON MATHEMATICAL SOCIETY.

FINANCIAL REPORT FOR THE SESSION 1891-2 (NOV. 12TH, 1891, TO NOV. 10TH, 1892).

Dr.	CASH ACCOUNT.	Cr.
£. s. d.		£. s. d.

General Fund.

	£.	s.	d.		£.	s.	d.
Cash at Bank	91	4	4	6 Subscriptions for 1892–3 paid in advance ...	6	6	0
130 Subscriptions owing—				7 Subscriptions struck off as irrecoverable—			
1 for 1887–8 ... ... £1 1 0				1 for 1887–8 ... ... 1 1 0			
4 for 1888–9 ... ... 4 4 0				1 for 1888–9 ... ... 1 1 0			
17 for 1889–90 ... ... 17 17 0				2 for 1889–90 ... ... 2 2 0			
30 for 1890–1 ... ... 31 10 0				2 for 1890–1 ... ... 2 2 0			
78* for 1891–2 ... ... 81 18 0				1 for 1891–2 ... ... 1 1 0			
	136	10	0		7	7	0
				Balance	214	1	4
	£227	14	4		£227	14	4

* Viz., 123, the number of Subscribers for 1891–2, less 5 Subscriptions paid in advance in 1890-1, and 40 paid in 1891–2.

De Morgan Medal Fund.

	£.	s.	d.		£.	s.	d.
Cash at Bank	7	4	2	Balance	7	4	2

CAPITAL ACCOUNT.

Sum Invested.	Description of Investment

Financial Report for the Session 1891–2 (Nov. 12th, 1891, to Nov. 10th, 1892).

CASH ACCOUNT.

Dr.

	£	s.	d.	£	s.	d.
Balance from 1890–91:—						
General Fund	£50	14	9			
De Morgan Medal Fund ...	4	6	6			
				55	1	3
Interest on Capital—						
Lord Rayleigh's Fund ...	57	17	10			
Lieut.-Col. Campbell's Fund	15	0	4			
Life Compositions Fund ...	28	15	10			
Invested Surplus Fund ...	13	6	8			
De Morgan Medal Fund ...	2	17	8			
				117	18	4
13 Entrance Fees				13	13	0
120 Subscriptions—						
3 for 1887–8	£3	3	0			
7 for 1888–9	7	7	0			
18 for 1889–90	18	18	0			
46 for 1890–1	48	6	0			
40 for 1891–2	42	0	0			
6 for 1892–3	6	6	0			
				126	0	0
Sales of *Proceedings*				74	1	6
Sale of Old Books				4	0	0
4 Life Compositions...				42	0	0
				£432	14	1

Cr.

	£	s.	d.	£	s.	d.
Printing *Proceedings*, &c.				222	5	0
Purchase of Journals, &c.				9	19	0
Binding				1	12	0
Postage, Stationery, and Sundries				17	10	0
Rent (including Coals and Gas)				19	9	0
Attendance—						
Mr. Stewardson	5	5	0			
Miss Hughes...	£4	14	0			
				9	19	0
Legacy duty on Dr. Hirst's bequest				7	0	0
Law charges				3	8	1
Diplomas for Foreign Members				1	3	6
Life Compositions Fund—						
Purchase of £32. 16*s*. 2*d*. 2¾ per Cent. Consols				31	10	0
Balance at Bank—						
General Fund	£91	4	4			
Life Compositions Fund	10	10	0			
De Morgan Medal Fund	7	4	2			
				108	18	6
				£432	14	1

Audited and found correct,

November 14*th*, 1892. (Signed) GEORGE HEPPEL.

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XXIV.

TWENTY-NINTH SESSION, 1892–93.

November 10*th*, 1892.

Annual General Meeting, held at 22 Albemarle Street, W.

Prof. GREENHILL, F.R.S., President, in the Chair.

Miss P. G. Fawcett was admitted into the Society.

The Treasurer (Mr. A. B. Kempe) read his Report. Its reception was moved by Mr. S. Roberts, seconded by Lt.-Col. Cunningham, and carried unanimously.

At the request of the Chairman, the meeting elected Mr. Heppel to act as Auditor, if he should be able to undertake the office.

From the Report of the Secretaries, it appeared that the number of the members during the Session had been 212, now reduced to 207, by the deaths of Drs. Hirst and Wolstenholme, and by the retirement of three members.

The number of compounders is 88.

The Society had to regret the loss, by death, of Prof. Enrico Betti, who was elected a foreign member December 14th, 1871; of Dr. Leopold Kronecker, who was elected a foreign member January 14th, 1875; of Dr. Thomas Archer Hirst, F.R.S., who was an original member of the Society; and of Prof. Joseph Wolstenholme, Sc.D., who was elected a member February 9th, 1871.*

* A detailed memoir of Prof. Kronecker, by M. Hermite, appeared in the *Comptes Rendus* for 4th January, 1892, and an interesting sketch is given by Prof. H. B. Fine in "Kronecker and his Arithmetical Theory of the Algebraic Equation," see *Bulletin of the New York Mathematical Society*, Vol. I., No. 8 (May, 1892); *cf.* also Vol. II., No. 3, p. 50. Notices of Dr. Hirst will be found in the *Biograph*, Vol. VI., and in *Nature*, Vol. XLV., p. 399. A brief sketch of Dr. Wolstenholme's career appeared in the "Obituary" of the *Times*.

The following communications had been made or received :—

On Selective and Metallic Reflection : Mr. A. B. Basset.

On the Classification of Binodal Quartic Curves : Mr. H. M. Jeffery.

On a Class of Automorphic Functions : Mr. W. Burnside.

Note on the Motion of a Fluid Ellipsoid under its own Attraction : Prof. M. J. M. Hill.

Note on Finding the G-points of a given Circle with respect to a given Triangle of Reference : Mr. J. Griffiths.

On Clifford's paper "On Syzygetic Relations among the Powers of Linear Quantics" : Prof. Cayley.

Note on the Identity $4(x^p-1)/(x-1) = Y^2 \pm pZ^2$: Prof. G. B. Mathews.

On the Theory of Elastic Wires : Mr. A. B. Basset.

The Equations of Propagation of Disturbances in Gyrostatically Loaded Media, and of the Circular Polarization of Light : Mr. J. Larmor.

On the Contacts of Systems of Circles : Mr. A. Larmor.

Note on Dirichlet's Formula for the Number of Classes of Binary Quadratic Forms for a Complex Determinant : Prof. G. B. Mathews.

Some Theorems relating to a System of Coaxal Circles : Mr. R. Lachlan.

On the Logical Foundations of Applied Mathematical Sciences : Mr. E. T. Dixon.

Note on the Inadmissibility of the usual reasoning by which it appears that the Limiting Value of the Ratio of two Infinite Functions is the same as the Ratio of their First Derived, with instances in which the result obtained by it is erroneous : Mr. E. P. Culverwell.

On Saint-Venant's Theory of the Torsion of Prisms : Mr. Basset.

The Simplest Specification of a given Optical Path, and the Observations required to determine it : Mr. J. Larmor.

On the Form of Hyper-elliptic Integrals of the First Order, which are Expressible as the Sum of Two Elliptic Integrals : Mr. W. Burnside.

On the Analytical Theory of the Congruency : Prof. Cayley.

Notes on Dualistic Differential Transformations : Mr. E. B. Elliott.

On certain Quartic Curves of the Fourth Class, and the Porism of the Inscribed and Circumscribed Polygon : Mr. R. A. Roberts.

Second Note on a Quaternary Group of 51840 Linear Substitutions : Dr. G. G. Morrice.

Note on the Skew Surfaces applicable upon a given Skew Surface : Prof. Cayley.

A Newtonian Fragment relating to Centripetal Forces : Mr. W. W. Rouse Ball.

Applications of a Theory of Permutations in Circular Procession to the Theory of Numbers : Major MacMahon.

The Harmonic Functions for the Elliptic Cone : Mr. E. W. Hobson.

Researches in the Calculus of Variations : Mr. E. P. Culverwell.

On an Operator which produces all the Covariants and Invariants of any System of Quantics : Mr. W. E. Story.

Note on the Algebraic Theory of Elliptic Transformation : Mr. J. Griffiths.

Further Note on Automorphic Functions : Mr. W. Burnnside.

Note on Approximate Evolution : Mr. H. W. Lloyd Tanner.

A Proof of the Exactness of Cayley's Number of Seminvariants of a given Type: Mr. E. B. Elliott.

The Second Discriminant of the Ternary Quantic $x'U+y'V+z'W$: Mr. J. E. Campbell.

On the Reflection and Refraction of Light from a Magnetized Transparent Medium: Mr. Basset.

The same journals had been subscribed for as in the preceding session. An additional exchange of *Proceedings* had been made, with the Mathematical Society of Edinburgh.

The Library had been enriched by a valuable bequest of books from the late Dr. Hirst.

The meeting next proceeded to the election of the new Council.

The Scrutators (Prof. Hudson and Mr. Ralph Holmes), having examined the balloting lists, declared the following gentlemen duly elected:—Mr. A. B. Kempe, F.R.S., President; Mr. A. B. Basset, F.R.S., Mr. E. B. Elliott, F.R.S., Prof. Greenhill, F.R.S., Vice-Presidents; Dr. J. Larmor, F.R.S., Treasurer; Messrs. M. Jenkins and R. Tucker, Secretaries. Other Members of the Council:—Mr. H. F. Baker, Dr. A. R. Forsyth, F.R.S., Dr. J. W. L. Glaisher, F.R.S., Mr. J. Hammond, Prof. M. J. M. Hill, Dr. E. W. Hobson, Mr. A. E. H. Love, Major MacMahon, R.A., F.R.S., and Mr. J. J. Walker, F.R.S.

The new President, having taken the chair, at once called upon Prof. Greenhill to read his Valedictory Address, of which the title was "Collaboration in Mathematics." On the motion of Major MacMahon, seconded by Prof. M. J. M. Hill, Prof. Greenhill was requested to allow his address to be printed in the *Proceedings*.

The following further communications were made:—

Some Properties of Homogeneous Isobaric Functions: Mr. E. B. Elliott.

On certain General Limitations affecting Hypermagic Squares: Mr. S. Roberts.

Note on Secondary Tucker-Circles: Mr. J. Griffiths.

Note on the Equation $y^2 = x^5 - x$: Prof. W. Burnside.

On a Group of Triangles inscribed in a given Triangle ABC, whose Sides are Parallel to Connectors of any Point P with A, B, C: Mr. R. Tucker.

A Note on Triangular Numbers: Mr. R. W. D. Christie.

The following presents were received :—

"Proceedings of the Edinburgh Mathematical Society," Vol. x.; 1892.

"Proceedings of the Royal Society of Edinburgh," Vol. xviii.; Session 1890–91.

"Jahrbuch über die Fortschritte der Mathematik," Band xxi., Heft 3; 1892.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 4e Serie, Tome ii.; 1891.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. v., No. 2.

"Proceedings of the Royal Society," Vol. li., No. 314; Vol. lii., No. 315.

"Proceedings of the Physical Society of London," Vol. xi., Part 4; October, 1892.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. xi., No. 1; Coimbra, 1892.

"Bulletin of the New York Mathematical Society," Vol. ii., No. 1.

"Royal Dublin Society—Scientific Proceedings," Vol. vii., Parts 3, 4.

"Royal Dublin Society—Scientific Transactions," Vol. iv., Parts 9–13.

"Nyt Tidsskrift for Mathematik," A. Tredje Aargang, Nos. 4, 5, 6; Copenhagen, 1892.

"Nyt Tidsskrift for Mathematik," B. Tredje Aargang, No. 3; Copenhagen, 1892.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," 1892, xxvi.–xl.

"Annali di Matematica," Tomo xx., Fasc. 2, 3.

"Educational Times," November, 1892.

"Journal für die reine und angewandte Mathematik," Band cx., Heft 3.

"Annals of Mathematics," Vol. vi., No. 7; June, 1892.

"Journal de l'Ecole Polytechnique," Cahier 61, 1891, and 62; 1892, Paris.

"Cambridge Philosophical Society—Transactions," Vol. xv., Pt. 3.

"Cambridge Philosophical Society—Proceedings," Vol. vii., Pt. 6.

"Annales de l'Ecole Polytechnique de Delft," Tome vii., 1891, Livr. 4; Leide, 1892.

Rayet (Mons. G.).—"Observations Pluviométriques et Thermométriques, faites dans le Département de la Gironde, de Juin 1890 à Mai 1891"; 8vo, Bordeaux, 1891.

Aiyar (S. Radhakrishna).—"Leaves from a Manual of Arithmetic"; 8vo, Madras, 1892.

"Invention," Vol. xiv., 703, 704.

"Atti della Reale Accademia dei Lincei," Vol. i., Fasc. 6, 7; Roma, 1892.

"Bollettino delle Pubblicazione Italiane ricevute per Diritto di Stampa," 1892, Nos. 163, 164.

"Traité Elémentaire des Quaternions," par P. G. Tait, translated by G. Plarr. (2 vols.; Paris, 1884.)

"Wiskundige Opgaven met de Oplossingen. Door de Leden van het Wiskundig Genootschap," vij.de Deel, 2de Stuk–5de Stuk; Amsterdam, 1891–92.

"Nieuw Archief voor Wiskunde," Deel xix.; Amsterdam, 1892.

Collaboration in Mathematics. (*Valedictory Address.*) *By* Prof. A. G. Greenhill. Read November 10th, 1892.

On the occasion of being admitted to this Society, it was my privilege to be present at the address delivered by Professor Stephen Smith, when he retired from the office of President.

Professor Smith at the outset asked permission to confine himself entirely to the history of Pure Mathematics, on the plea that his immediate successor, Lord Rayleigh, in looking round in his turn for a subject, would be attracted by the domain of Applied Mathematics, of which, as Professor Smith remarked, he is so well fitted to take a comprehensive view.

My predecessor, Mr. J. J. Walker, selected as his subject "The Influence of Applied on the Progress of Pure Mathematics," and he showed us how many of the most abstruse theorems of pure analysis owe their origin to ideas, which arose in connexion with concrete and even practical requirements.

The developments of Pure Mathematics, and of Applied Mathematics, and the influence of Pure and Applied Mathematics on each other, having thus received such eloquent and exhaustive treatment from my predecessors, I was compelled to turn elsewhere for a subject suitable for a short valedictory address.

A conference of mathematicians was to be held last summer at Nuremberg; and there, I thought, would be the opportunity of meeting, in conclave and in society, the leading mathematicians of the day. It would interest you, I hoped, to hear a short account of their methods and ideas, and perhaps more, to learn what they thought of us in return.

But, unfortunately for these plans, the state of quarantine in Germany caused the Nuremberg meeting to be postponed till next year; so I am obliged to come before you this evening in the condition of one who has to make shift with a few general remarks on the state of our subject, with some special reference to Collaboration in Mathematics.

But in the first place, we must congratulate those members who have worked so efficiently to fill up the gaps pointed out by Professor Smith, in the subjects requiring advanced text-books.

At the time Professor Smith made this observation he could only instance Cayley's *Elliptic Functions;* but at the present day we possess (or shall shortly possess), as works of our members, advanced treatises on the theory of numbers, of equations, of functions and differential equations, on advanced trigonometry, analytical statics, hydrodynamics, optics, and elasticity. The authors' names are familiar to you; and we hope that other treatises are in course of preparation, to gratify acknowledged needs.

A large part of Professor Smith's address was devoted, even if indirectly, to the subject of Elliptic Functions, in consequence of its important bearing on all the domain of Pure Analysis; and it is interesting at this date to reflect on the vicissitudes of its cultivation, as a branch of academic study, by the light of the remarks on this point in Dr. Glaisher's address.

In my own case I have been attracted to this subject, from the point of view of Mr. Walker's address, by looking at the theorems as they arise in the solution of definite mechanical problems.

Even in this way some of its most abstruse analytical theorems receive new interest; to mention an instance, the treatment by Abel of the *pseudo-elliptic integrals*, as derived from the expression of the square root of a quartic in a periodic continued fraction.

I am not aware whether it has been pointed out, that in Abel's method we assign an arbitrary value to the elliptic function of an aliquot part of a period, and thence determine the corresponding value of the modulus; in this way the degree of the equation requiring solution is considerably reduced. Thus, for trisection, the degree is reduced from four to unity; for quinquisection, from six to two; for division by seven, from eight to three; and so on.

I should like, with your permission, to indicate briefly these results on the board.

Abel's integrals, in the form (*Œuvres Complètes*, t. II., p. 162)

$$I = (n+2)\int \frac{x+k}{\sqrt{X}}\,dx,$$

where X is a quartic, can be reduced, when we know a factor $x-a$ of X, to the Weierstrassian form

$$I = \int \frac{Pz+(n+2)\,\delta}{z\sqrt{Z}}\,dz,$$

by the substitution $$x-a = \frac{M}{z};$$

[redacted]

and now the cubic Z can be made to assume the form

$$Z = 4z\,(\alpha z-\beta)^2-(\gamma z-\delta)^2,$$

so that the roots of $Z=0$ are the squares of the roots of a simpler cubic

$$2\alpha t^3-\gamma t^2-2\beta t+\delta = 0\,;$$

also $z=0$ corresponds to the parameter of this pseudo-elliptic integral of the third kind.

When n is odd, the result can be written in the form

$$I = 2\cos^{-1}\frac{z^{\frac{1}{2}(n-3)}+Bz^{\frac{1}{2}(n-5)}+\dots}{2\alpha z^{\frac{1}{2}n}}\sqrt{Z}$$

$$= 2\sin^{-1}\frac{Pz^{\frac{1}{2}(n-1)}+Qz^{\frac{1}{2}(n-3)}+\dots}{2\alpha z^{\frac{1}{2}n}}.$$

It will be noticed that the knowledge of a factor of the quartic X has the effect of reducing the degree in the results to one half that given by Abel.

Now, in the case of trisection,

$$n+2=3\,;$$

we find $\beta=0$, and we can put $\alpha=1$, $\gamma=1$; and then Klein's parameter τ is connected with δ by the linear equation

$$\tau = -\frac{4-27\delta}{27\delta}\,;$$

also $\quad J : J-1 : 1 = (\tau-1)(9\tau-1)^3 : (27\tau^2-18\tau-1)^2 : -64\tau$

(Klein, "On the Transformation of the Elliptic Functions," *Proc. Lond. Math. Soc.*, Vol. IX.).

We may now write Abel's pseudo-elliptic integral, for this case of trisection, in the form

$$I = \int\frac{z-3\delta}{z\sqrt{\{z^3-(z-\delta)^2\}}}\,dz$$

$$= 2\sin^{-1}\frac{\sqrt{\{z^3-(z-\delta)^2\}}}{z^{\frac{3}{2}}} = 2\cos^{-1}\frac{z-\delta}{z^{\frac{3}{2}}}.$$

Quinquisection: $n+2=5$;

here we find, from Abel's results, we can put

$$\alpha=1,\quad \beta=c,\quad \gamma=c-1,\quad \delta=-c;$$

and now Klein's $$\tau=c+11-\frac{1}{c},$$

a quadratic for c when τ is given, corresponding to the parameters

$$\tfrac{2}{5}\omega \quad\text{and}\quad \tfrac{4}{5}\omega,$$

ω denoting a period (the imaginary period, according to our result, where the integral is circular).

Here $$J : J-1 : 1$$

$$=(\tau^2-10\tau+5)^3 : (\tau^2-22\tau+125)(\tau^2-4\tau-1)^2 : -1728\tau.$$

On comparison with Halphen's equations (*Fonctions Elliptiques*, t. III., p. 3), we find

$$12g_2=1-12c+14c^2+12c^3+c^4,$$

$$216g_3=(1+c^2)(1-18c+74c^2+18c^3+c^4),$$

$$\Delta=-c^5\left(c+11-\frac{1}{c}\right);$$

and $$t=c^2$$

is a root of Halphen's equation (9), p. 5,

$$5t^6-12g_2t^5+10\Delta t^3+\Delta^2=0.$$

Also $$x=-\tfrac{1}{6}(1+c^2)$$

is a root of Halphen's equation (4), p. 3,

$$x^6-5g_2x^4-40g_3x^3-5g_2^2x^2-8g_2g_3x-5g_3^2=0.$$

We can now write Abel's pseudo-elliptic integral for this case in the form

$$I=\int\frac{(c+3)\,z-5c}{z\sqrt{\left[4z\,(z-c)^2-\{(c-1)\,z+c\}^2\right]}}\,dz$$

$$=2\cos^{-1}\frac{(z-1)\sqrt{Z}}{2z^{\frac{1}{2}}}$$

$$=2\sin^{-1}\frac{(c+3)\,z^2-(2c+1)\,z+c}{2z^{\frac{3}{2}}}.$$

7-section: $n+2 = 7.$

We find, from Abel's results,

$$\alpha = 1, \quad \beta = -c^2-c^3, \quad \gamma = 1+3c+c^2, \quad \delta = -c^3-c^4;$$

and here
$$\tau = -\frac{1+8c+5c^2-c^3}{c(1+c)},$$

a cubic equation for c, corresponding to the parameters

$$\tfrac{2}{7}\omega, \quad \tfrac{4}{7}\omega, \quad \tfrac{6}{7}\omega;$$

and
$$J : J-1 : 1 = (\tau^2+13\tau+49)(\tau^2+5\tau+1)^3$$
$$: (\tau^4+14\tau^3+63\tau^2+70\tau-7)^2 : 1728\tau.$$

Here
$$t = -c^4(1+c)^4$$

is a root of Halphen's equation (59), p. 75, *F. E.*, t. III.,

$$7t^8-2^8.3^3g_2t^7-2.5.7\Delta t^6-3^2.7\Delta^2t^4-2.7\Delta^3t^2-\Delta^4 = 0;$$

while
$$x = -\tfrac{1}{4}(1+c+c^2)^2$$

is a root of equation (15), p. 51,

$$x^8-21g_2x^6-2.3^3.7g_3x^5 \ldots\ldots = 0.$$

Also
$$12g_2 = (1+c+c^2)(1+11c+30c^2+15c^3-10c^4-5c^5+c^6),$$
$$\Delta = -c^7(1+c)^7(1+8c+5c^2-c^3);$$

and Abel's corresponding pseudo-elliptic integral may be written

$$I = \int \frac{(1+3c-3c^2)z+7(1+c)}{z\sqrt{\left[4cz(cz+1+c)^2-\{(1+3c+c^2)z+1+c\}^2\right]}}\,dz$$

$$= 2\cos^{-1}\frac{z^2+(2-c)z+1}{2c^{\frac{3}{2}}z^{\frac{3}{2}}}\sqrt{Z}$$

$$= 2\sin^{-1}\frac{(1+3c-3c^2)z^3+(3+4c-3c^2+c^3)z^2+(3+2c-2c^2)z+1+c}{2c^{\frac{3}{2}}z^{\frac{3}{2}}}.$$

9-section: $n+2 = 9.$

We find we can put

$$\alpha = (1-c)^2, \quad \beta = -c+c^2-c^3, \quad \gamma = 1-2c+c^3+c^4, \quad \delta = \beta;$$

and now Kiepert's parameter ξ (*Math. Ann.*, XXXII., p. 66) is connected with c by the relation

$$\xi = -\frac{1+3c-6c^2+c^3}{c(1-c)},$$

a cubic equation for c, corresponding to the parameters

$$\tfrac{2}{9}\omega, \quad \tfrac{4}{9}\omega, \quad \tfrac{8}{9}\omega;$$

also $\quad J : J-1 : 1 = (\xi+3)^3(\xi^3+9\xi^2+27\xi+3)^3$

$$: (\xi^6+18\xi^5+135\xi^4+504\xi^3+891\xi^2+486\xi-27)^2$$

$$: 1728\xi(\xi^2+9\xi+27).$$

Abel's corresponding pseudo-elliptic integral is

$$I = \int \frac{Pz+9\delta}{z\sqrt{Z}}\,dz$$

$$= 2\cos^{-1}\frac{z^3+Bz^2+Cz+D}{2(1-c)^2 z^{\frac{9}{2}}}\sqrt{Z}$$

$$= 2\sin^{-1}\frac{Pz^4+Qz^3+Rz^2+Sz+T}{2(1-c)^2 z^{\frac{9}{2}}},$$

where $\quad B = -3(2-2c+c^2),$

$$C = (1-c+c^2)(5-3c+c^2),$$

$$D = -(1-c+c^2)^2,$$

$$P = 7-18c+15c^2-5c^3,$$

$$Q = -14+53c-76c^2+61c^3-25c^4+5c^5,$$

$$R = (1-c+c^2)(7-33c+38c^2-23c^3+6c^4-c^5),$$

$$S = -(1-c+c^2)^2(1-9c+6c^2-2c^3),$$

$$T = -c(1-c+c^2)^3.$$

11-section: $\quad n+2 = 11.$

Here I have found that, in Abel's notation (*Œuvres Complètes*, t. II., p. 162), with $q_0 = 0$, or $q_4 = q_5$, then, if

$$q(1+q) = c(1+c)^2,$$

$$q_4 = q_5 = 2c(1+c+q);$$

and we may put $\quad \alpha = 1, \quad \beta = (c^2+c^3)q,$

$$\gamma = 1+2c-c^3+(2+c)q, \quad \delta = -(c^2+c^3)^2 q.$$

A quintic relation is now to be expected, corresponding to the parameters

$$\tfrac{2}{11}\omega,\quad \tfrac{4}{11}\omega,\quad \tfrac{6}{11}\omega,\quad \tfrac{8}{11}\omega,\quad \tfrac{10}{11}\omega,$$

connecting c with the parameter ξ employed by Kiepert (*Math. Ann.*, XXXII., p. 93), or with the parameter τ of Klein (Klein und Fricke, *Vorlesungen über die Theorie der Elliptischen Modulfunctionen*, II., p. 437, &c.); but I have not yet succeeded in determining this relation, and I am relying upon the assistance of others.

Similar simple results hold for the even values of n,

$$2,\ 4,\ 6,\ 8,\ \ldots;$$

but now Abel's quartic X is found to break up into two quadratic factors X_1 and X_2, and the Weierstrassian form is unsuitable.

We can, however, express the result, if logarithmic or hyperbolic, in the form

$$I = (n+2)\int \frac{x+k}{\sqrt{(X_1X_2)}}\,dx$$

$$= 2\cosh^{-1}\left(Ax^{\frac12 n}+Bx^{\frac12(n-2)}+\ldots\right)\sqrt{X_1}$$

$$= 2\sinh^{-1}\left(Px^{\frac12 n}+Qx^{\frac12(n-2)}+\ldots\right)\sqrt{X_2}.$$

Thus, for 4-section: $\qquad n+2=4;$

we find
$$X_1 = x^2+(1+2c)\,x+c^2,$$
$$X_2 = x^2+(1-2c)\,x+c^2;$$

$$I = 2\sinh^{-1}\frac{x+1-c}{2c^{\frac12}}\sqrt{X_1} = 2\cosh^{-1}\frac{x+1+c}{2c^{\frac12}}\sqrt{X_2}.$$

6-section: $\qquad n+2=6;$

$$X_1 = x^2+(1+c)^2\,x+c^2\,(1+c)^2,$$
$$X_2 = x^2+(1-c)^2\,x+c^2\,(1-c)^2;$$

$$I = 2\sinh^{-1}\frac{x^2+(2-c+c^2)\,x+(1-c)(1-c^3)}{2c^{\frac12}\,(1-c^2)}\sqrt{X_1}$$

$$= 2\cosh^{-1}\frac{x^2+(2+c+c^2)\,x+(1+c)(1+c^3)}{2c^{\frac12}\,(1-c^2)}\sqrt{X_2};$$

and so on; the results being of half the degree given by Abel, in consequence of the knowledge of the factors X_1 and X_2.

From Abel's results we are able to construct a series of solvable cases of the complicated motion assumed by a body under no forces,

by a top or gyrostat, or by a rotating projectile in a surrounding fluid medium; as indicated in Halphen's *Fonctions elliptiques.*

Mr. Burnside tells me he has discovered an application of the pseudo-elliptic integrals to the flow of electricity on an *anchor-ring*, when the differences of the longitude and quasi-latitude of the electrodes are aliquot parts of the circumference; and then the current itself divides into parts in a simple ratio.

In Abel's method it is not necessary (as in Jacobi's treatment, *Crelle*, Bd. VII., *Gesammelte Werke*, Bd. I., p. 329) to consider the resolution of the quartic X under the radical; and now the parameter of the corresponding elliptic integral of the third kind is an aliquot part of one of the periods.

But, in the applications to dynamical problems, the parameter has an added quarter or half period, implying that the resolution of the quartic has been effected; and this requirement introduces a fresh difficulty, which may in competent hands lead to important developments of Abel's method.

To illustrate these curious analytical results which may thus arise, requiring the resources of pure analysis, I may mention that, in constructing such solvable cases, depending, in the language of elliptic functions, on the division of a period by seven, a cubic equation ($Z = 0$) turns up requiring solution, which may be illustrated geometrically by a certain ternary homogeneous equation representing a non-unicursal cubic curve.

Professor Smith mentions that the solution in integers of such an equation depends on some criterion not yet discovered; although, as he says, it ought not to lie beyond the present scope of analysis.

But, as the Theosophist would say, "this is a detail of which our mathematicians know so little."

Mr. G. B. Mathews is fortunately interested at the present time in Abel's general theory, and I have great hopes that he will succeed in this problem, and at the same time present a general theory.

This episode encourages me to make the suggestion that the modern principle of the Subdivision of Labour might with advantage be employed in mathematical research; and that by Collaboration on new ideas our progress would be accelerated.

Prof. Tait makes a feeling complaint somewhere that in looking back at the time spent on working out a new development, the greater part of it is occupied on analysis that might well be turned over to a trained mathematical student.

Most of us, I expect, will echo Prof. Tait's opinion; but, so far as I

am aware, the only attempt at the present day to introduce collaboration into mathematical work is to be found in Prof. Klein's *Seminär* at Gottingen.

In his wonderful lectures on the ikosahedron, modular, hyper-elliptic, and Abelian functions, Prof. Klein reserves his powers for a general comprehensive view of the subject; and the detailed algebraical analytical development is handed over to his pupils to work up, at the same time stimulating their interest.

The system appears to work very well, judging by the published results in the *Mathematische Annalen;* as in all scientific work, there is no pretence at concealment.

We are not like the artist, say Rubens, or the writer, say Dumas, who are reported to have passed off, as their own original work, the parts elaborated by pupils and assistants.

An advantage of such collaboration lies in the admission of the outside world to the order of development of new ideas. We see the structure growing, and can trace the scaffolding employed in its erection.

The perfect geometrical form of the *Principia* was for a long time a hindrance to the formation of a school in this country to follow up Newton's discoveries, as no successors arose who could wield this great weapon of geometry with facility. But the communications brought before this Society by Mr. Ball and Dr. Glaisher, on the rough notes of Newton's calculations, preserved in his manuscripts, are of great value and encouragement to us, in showing that Newton himself, for purposes of discovery, adopted an analytical method similar to that we employ nowadays; and that the form of the results was subsequently recast in the Greek geometrical style.

I presume we are all agreed, however, as to the value of a study of the *Principia*, as a chastening influence on the luxuriance of symbols and analysis. For my part I should prefer to see the whole three books prescribed in the Cambridge course, to be studied in the original Latin.

In two hundred years no real advance has been made to complete the speculations as to the cause of gravity with which the *Principia* concludes; nor has any exception been observed, in the solar system, to Kepler's Laws and the Law of Gravitation.

The first report of the newly discovered satellite of Jupiter gave an incompatible distance and period; these numbers have been subsequently corrected; but any discrepancy, if confirmed, would lead to

most interesting speculations as to the causes modifying the universal applicability of Newton's Law.

"Heresy in science is the author of all progress," and to discover and theorize on the smallest discrepancies with simple acknowledged rules is certain to lead to important development.

In Prof. Smith's words, "an apparent contradiction (as distinct from a mere misunderstanding) is always to be regarded as an indication of some undiscovered truth."

In Electricity, the "law of the inverse square" is capable of indirect experimental verification with great accuracy; but we were indebted to Maxwell, in the Mathematical Tripos of 1877, for a very beautiful question, of a somewhat heretical nature, in which he showed that, if we assume the law to vary as the negative power $(2+p)$, where p is small, the value of p could be inferred by means of a formula from a simple electrical experiment. So far the value of p has been found insensible; but it will be interesting, when more detailed reports are to hand, to employ Maxwell's method on Jupiter's new satellite.

Newton's Third Law, that "action and reaction are equal and opposite"—according to Maxwell's interpretation, the definition of a *stress*—implies in its generality that the inertia of the medium by which the stress is propagated is insensible.

Maxwell, in his *Electricity and Magnetism*, acting as the interpreter of Faraday's views, first taught us to look for the explanation of electrical phenomena in the stress in the medium or dielectric which propagates the action to a distance.

The effect of quasi-electrical inertia, as exhibited in the Herz oscillations, has only recently attracted attention; and perhaps we shall find that the corresponding inertia of space, in which gravity is propagated, may either slightly modify the laws of gravity, or, on the other hand, be the means of accounting for their existence.

Formerly, the mathematician, when called upon to justify the utility of his subject, would point to its astronomical achievements, if only for their bearing on Navigation.

Nowadays I think we could instance the applications of Electricity, as the creation of men whose genius was essentially mathematical.

Much of the theory which served in physical astronomy was found to be immediately applicable in electrical science; and now, in return, it seems that the minute manifestations of electricity will help to throw light on the mysterious laws which govern the motion of our universe.

Returning to the question of Collaboration, the correspondence of

Newton and Cotes, we find, is principally concerned with the preparation of a new edition of the *Principia*. The laborious part of the work was undertaken by Cotes, and it seems a pity that Newton did not employ him as a collaborator earlier.

Mr. Ball read to us an interesting paper on "Newton's Classification of Cubic Curves," in which he showed us that most of the theorems rediscovered of late years were already known to Newton.

In my opinion, Newton should, for his genius, have been otherwise occupied; but doubtless he became fascinated, in a manner with which I presume we are most of us familiar, with the algebraical details of some calculations in connexion with an astronomical or dynamical question, while the great theory of gravitation was taking shape in his mind.

So, too, Maxwell informs us that he was lured away into the *Analysis Situs*, from being compelled to develop it for his electromagnetic theory; and even Rankine, the mathematician of the engineer, confesses that he was once attracted by the theory of numbers, and wasted much time over it, most agreeably, no doubt, but to no purpose. "Le temps le mieux employé est celui que l'on perd."

During the past year the Society has suffered great loss, in Dr. Hirst, Professor Wolstenholme, and two foreign members, Prof. Kronecker and Signor Betti.

Dr. Hirst was an original member of our Society, and its first Treasurer; he took great interest in our meetings, and he filled the office of President. The Society is indebted to him for the bequest of his valuable library.

His work on the *Correlation of Space* brought him into correspondence with the Italian mathematicians of the school of Cremona; he contributed often to foreign journals, in which his linguistic knowledge showed to advantage.

Signor Betti has favoured us with communications on the subject of Elasticity, in which the principles are presented in an elegant and condensed form.

Professor Wolstenholme's mathematical fame rests on his wonderful collection of original mathematical problems, a production peculiarly characteristic of the British school. The influence of these problems on the formation of the taste and ability of a student cannot be overestimated. It was my good fortune to be his colleague for some time at Cooper's Hill, and to keep up a continuous acquaintance.

Prof. Kronecker's work, chiefly concerned with the Theory of

Numbers, is too great and extensive to receive more than mere mention here; for details we must refer to a memoir of him in the *Comptes Rendus*, January, 1892, written by M. Hermite.

Looking through the communications to the Society in the past two years, we find papers on all branches of mathematical research; and it would be invidious to single out any one for special commendation, when we consider the different tastes and proclivities of those who cultivate our science, arising from the specialization which is now a necessity on our part.

Speaking for myself, however, I must express the pleasure I experienced in reading Mr. Burnside's communications on "Functions derived from their Discontinuities," and on "Automorphic Functions." In these he works out the purely analytical developments of Klein, Weber, and Poincaré, with reference to the physical standpoint of electrical and hydrodynamical applications. This physical interpretation, to my mind, throws light and interest on the complicated results which arise, and gives a hold by which the mind can obtain a first grasp of the subject.

In conclusion I should like to say a few words about our Society.

In the twenty-seven years of its existence we can look back upon the active membership of those mathematicians who will be remembered by posterity as the leaders of thought of this generation; and the London Mathematical Society has already amply justified its existence, in the works of past members, such as De Morgan, Spottiswoode, Stephen Smith, Hirst, Maxwell, Wolstenholme, Clifford, and Buchheim.

I think we may congratulate ourselves on its present condition intellectually, financially (thanks to our President), and numerically.

It has been said that we can never be a numerous body; but, by judicious recruiting, our numbers could doubtless be still further added to.

I leave it to our junior members to come forward in this work, and also in otherwise keeping up the vitality of our subject and Society; so that posterity may hold the same high opinion of them as we do of our predecessors.

Note on the Equation $y^2 = x(x^4-1)$. *By* W. Burnside.

Received November 1st, and read November 10th, 1892.

The Cartesian co-ordinates of a point on a curve whose deficiency is greater than one, cannot be expressed as rational or as elliptic functions of a single parameter; but it has recently been shown that, in a large number of cases, including certainly all curves whose equations are of hyper-elliptic form with real coefficients, the co-ordinates of a variable point on the curve can be expressed as uniform functions of a single parameter with an infinite number of essentially singular points, the functions being automorphic with respect to a certain group of substitutions.

Although, however, the possibility of this mode of representing the coefficients has been proved, no instance has ever been given, so far as I know, in which the representation has actually been carried out for a particular equation. Indeed, to do this it would generally be necessary to find a group having the required relation to the equation, and there is, I believe, no known method of doing this. If, however, from independent considerations, the relation in question between a given equation and a known group has once been established, it will become a matter of calculation to carry out the process referred to.

In the case of the equation

$$y^2 = x(x^4-1) \qquad \text{(i.)},$$

the actual expressions for x and y, as functions of a single parameter, can, by such considerations as those just mentioned, be obtained with remarkably simple analysis, and the result is perhaps of sufficient interest to justify me in communicating it to the Society.

It is convenient to begin by giving two well-known formulæ in elliptic functions that will be required. These are

$$\wp 2u - e = \left[\frac{\wp^2 u - 2e\wp u - e^2 - e'e''}{\wp' u}\right]^2,$$

and

$$\wp(u+\omega) - e = \frac{(e-e')(e-e'')}{\wp u - e};$$

whence, by differentiation,

$$\frac{\wp'(u+\omega)}{\wp' u} = -\frac{(e-e')(e-e'')}{(\wp u - e)^2}.$$

If 2ω, $2\omega'$ are any pair of primitive periods of the elliptic functions, these formulæ are equivalent to

$$\wp 2u - \wp\omega = \left\{\frac{\left(\wp u - \wp\frac{\omega}{2}\right)\left[\wp u - \wp\left(\omega' + \frac{\omega}{2}\right)\right]}{\wp'(u)}\right\}^2 \quad \text{......(ii.)}$$

and
$$\frac{\wp'(u+\omega)}{\wp' u} = -\left\{\frac{\wp\frac{\omega}{2} - \wp\omega}{\wp u - \wp\omega}\right\}^2 \quad \text{..................(iii.).}$$

In Klein-Fricke, *Theorie der Elliptischen Modulfunctionen*, I., p. 652, it is shown that the x and y of equation (i.) are expressible as modular functions for that sub-group of the modular group which is formed of all substitutions

$$\omega, \left(\frac{\alpha\omega+\beta}{\gamma\omega+\delta}\right), \quad \alpha\delta - \beta\gamma = 1,$$

such that $\begin{pmatrix}\alpha, & \beta \\ \gamma, & \delta\end{pmatrix}$ is congruent to modulus 8 with one of the forms

$$\begin{pmatrix}1, & 0 \\ 0, & 1\end{pmatrix}, \quad \begin{pmatrix}1, & 4 \\ 4, & 1\end{pmatrix}, \quad \begin{pmatrix}5, & 4 \\ 0, & 5\end{pmatrix}, \quad \begin{pmatrix}5, & 0 \\ 4, & 5\end{pmatrix};$$

and moreover that x is then that modular function which Prof. Klein calls the octahedral irrationality. It does not, however, enter into the plan of Prof. Klein's work to exhibit x and y explicitly as functions of ω, which is the object of the present note.

There is no difficulty in showing that the twenty-four values of Prof. Klein's octahedral irrationality are given in terms of elliptic functions of submultiples of the periods by the expression

$$\frac{\wp\frac{\omega}{2} - \wp\omega}{\wp\frac{\omega'}{2} - \wp\omega},$$

and the twenty-three expressions derivable from it by the substitutions

$$\omega_1 = \omega + \omega', \quad \omega_1' = \omega'$$

and
$$\omega_2 = -\omega', \quad \omega_2' = \omega.$$

If now
$$x = \frac{\wp\frac{\omega}{2} - \wp\omega}{\wp\frac{\omega'}{2} - \wp\omega},$$

then, by (iii.), $$x^2 = -\frac{\wp'\left(\frac{\omega'}{2}+\omega\right)}{\wp'\frac{\omega'}{2}}.$$

The addition equation gives

$$\wp\omega'+2\wp\frac{\omega'}{2} = \frac{1}{4}\frac{\wp'^2\frac{\omega'}{2}}{\left(\wp\omega'-\wp\frac{\omega'}{2}\right)^2},$$

and $$\wp\omega'+2\wp\left(\frac{\omega'}{2}+\omega\right) = \frac{1}{4}\frac{\wp'^2\left(\frac{\omega'}{2}+\omega\right)}{\left[\wp\omega'-\wp\left(\frac{\omega'}{2}+\omega\right)\right]^2},$$

or since, as may be deduced at once from (ii.),

$$2\wp\omega' = \wp\frac{\omega'}{2}+\wp\left(\frac{\omega'}{2}+\omega\right),$$

$$\wp\omega'+2\wp\left(\frac{\omega'}{2}+\omega\right) = \frac{1}{4}\frac{\wp'^2\left(\frac{\omega'}{2}+\omega\right)}{\left[\wp\omega'-\wp\frac{\omega'}{2}\right]^2}.$$

Hence $$x^4 = \frac{\wp\omega'+2\wp\left(\frac{\omega'}{2}+\omega\right)}{\wp\omega'+2\wp\frac{\omega'}{2}},$$

and $$x^4-1 = 2\frac{\wp\left(\frac{\omega'}{2}+\omega\right)-\wp\frac{\omega'}{2}}{\wp\omega'+2\wp\frac{\omega'}{2}}$$

$$= 4\frac{\wp\omega'-\wp\frac{\omega'}{2}}{\wp\omega'+2\wp\frac{\omega'}{2}}$$

$$= 16\frac{\left(\wp\omega'-\wp\frac{\omega'}{2}\right)^2}{\wp'^2\frac{\omega'}{2}}.$$

Therefore $$x(x^4-1) = -16\,\frac{\left(\wp\frac{\omega}{2}-\wp\omega\right)\left(\wp\frac{\omega'}{2}-\wp\omega'\right)^2}{\left(\wp\frac{\omega'}{2}-\wp\omega\right)\wp'^2\frac{\omega'}{2}}.$$

Taking account of equation (ii.), this can be written in the form

$$x(x^4-1)$$
$$= -16\left\{\frac{\left(\wp\frac{\omega'}{2}-\wp\omega'\right)\left(\wp\frac{\omega'}{4}-\wp\frac{\omega'}{2}\right)\left[\wp\frac{\omega'}{4}-\wp\left(\frac{\omega'}{2}+\omega\right)\right]\times\left(\wp\frac{\omega}{4}-\wp\frac{\omega}{2}\right)\left[\wp\frac{\omega}{4}-\wp\left(\frac{\omega}{2}+\omega'\right)\right]}{\left(\wp\frac{\omega'}{4}-\wp\frac{\omega}{2}\right)\left[\wp\frac{\omega'}{4}-\wp\left(\frac{\omega}{2}+\omega'\right)\right]\wp'\frac{\omega}{4}\,\wp'\frac{\omega'}{2}}\right\}^2;$$

and therefore, finally, if

$$y = 4i\,\frac{\left(\wp\frac{\omega'}{2}-\wp\omega'\right)\left(\wp\frac{\omega'}{4}-\wp\frac{\omega'}{2}\right)\left[\wp\frac{\omega'}{4}-\wp\left(\frac{\omega'}{2}+\omega\right)\right]\times\left(\wp\frac{\omega}{4}-\wp\frac{\omega}{2}\right)\left[\wp\frac{\omega}{4}-\wp\left(\frac{\omega}{2}+\omega'\right)\right]}{\left(\wp\frac{\omega'}{4}-\wp\frac{\omega}{2}\right)\left[\wp\frac{\omega'}{4}-\wp\left(\frac{\omega}{2}+\omega'\right)\right]\wp'\frac{\omega}{4}\,\wp'\frac{\omega'}{2}},$$

and $$x = \frac{\wp\frac{\omega}{2}-\wp\omega}{\wp\frac{\omega'}{2}-\wp\omega},$$

then $$y^2 = x(x^4-1).$$

Taking account of homogeneity, x and y are thus expressed as uniform functions of the parameter ω'/ω, and they each have the real axis for an essentially singular line.

Some Properties of Homogeneous Isobaric Functions. By E. B. Elliott. Received and read November 10th, 1892.

1. The present paper is a sequel to one which I communicated to the Society at its last meeting (Vol. xxiii., pp. 298–304), entitled, "A Proof of the Exactness of Cayley's Number of Seminvariants of a Given Type." The two articles which immediately follow supply omissions in that paper. In the remaining articles the theorem on which my argument was based is transformed, and the result examined for its own sake without reference to the particular application.

2. Attention was in my former paper quite unnecessarily confined to a single binary quantic, or, as I would say by preference, to a single set of constituents $a_0, a_1, a_2, \ldots a_n$. The proof of the Cayley-Sylvester theorem as to the number of asyzygetic seminvariants of a given type of a system of binary quantics, or, say, in a system of sets of constituents, is precisely the same.

Let there be quantics of degrees $n, n', n'', \ldots$, with coefficients $(a_0, a_1, \ldots a_n)$, $(a'_0, a'_1, \ldots a'_{n'})$, $(a''_0, a''_1, \ldots a''_{n''})$, ... The number of products of whole weight w of given numbers $i, i', i'', \ldots$ of constituents chosen from these sets of coefficients respectively is denoted by

$$(w;\ i, n;\ i', n';\ i'', n'';\ \ldots).$$

Let now

$$\Omega \text{ denote } \Sigma\left(a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}+\ldots+na_{n-1}\frac{d}{da_n}\right) \quad\ldots\ldots\ldots\ldots(1),$$

$$O \quad " \quad \Sigma\left(na_1\frac{d}{da_0}+(n-1)\,a_2\frac{d}{da_1}+\ldots+a_n\frac{d}{da_{n-1}}\right) \ \ldots\ (2),$$

and $$\eta \quad " \quad \Sigma\,(in)-2w \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

the summations referring to all the sets of coefficients.

Exactly as in § 2 of the paper referred to, we have the known theorems that, when the operations are on any product as above, or on any gradient or linear function with constant coefficients of such products of the same type $w;\ i, i', i'', \ldots$,

$$\Omega O - O\Omega = \eta,$$

$$\Omega O^r - O^r\Omega = r\,(\eta - r + 1)\,O^{r-1};$$

and the reasoning from these is just as before. We obtain the same form of conclusion, viz., that

$$u = \Omega\left\{\frac{1}{1.\eta}\,O - \frac{1}{1.2.\eta.\eta+1}\,O^2\Omega + \frac{1}{1.2.3.\eta.\eta+1.\eta+2}\,O^3\Omega^2 - \dots\right\}u \qquad \text{.................}(4),$$

provided that $\eta > 0$; and the interpretation is that any product, or linear function of products of the type considered, can be produced by operation with Ω on a linear function of products of whole weight $w+1$, and the same partial orders $i, i', i'', \dots$ as before. Putting, then, w for $w+1$, the most general linear function of products of type w; $i, i', i'', \dots$, when operated on with Ω, yields the most general linear function of products of type $w-1$; $i, i', i'', \dots$, provided that $\Sigma(in) - 2(w-1) > 0$, *i.e.*, that $\Sigma(in) - 2w \not< -1$. Accordingly, in this case, the number of linearly independent seminvariants, linear functions which Ω annihilates, of the type w; $i, i', i'', \dots$, in the system of sets of coefficients, which is known to be at least

$$(w;\ i, n;\ i', n';\ i'', n'';\ \dots) - (w-1;\ i, n;\ i', n';\ i'', n'';\ \dots),$$

is exactly that number.

Another consequence of the generality of the gradient Ωu, when u is general and such that $\eta \not< -1$, is that in such a case the number thus found as a difference cannot be negative.

3. It ought not to have escaped my attention that operators of form like that in my theorem above, have presented themselves to and been used by Hilbert (*Mathematische Annalen*, Vol. xxx., pp. 15, &c., Vol. xxxvi., p. 523). He has used such operators, in fact, in a proof, different from mine, of the exactness of Cayley's formula, but does not seem to have noticed the fact (4) which is with me fundamental. He has proved that, η, the characteristic of v, being $\not< -1$,

$$\Omega\left\{1 - \frac{1}{1.\eta+2}\,O\Omega + \frac{1}{1.2.\eta+2.\eta+3}\,O^2\Omega^2 - \dots\right\}v = 0 \text{......}(5),$$

which might be produced from (4) above by putting Ωv for u, and consequently replacing η by $\eta+2$. His theorem then follows from mine; but the reverse does not seem to be the case, unless we assume that, v being general, Ωv is general; and this I was not at liberty to assume, for it is what my aim was to establish.

A particular result of (5) may here be noticed. Let $\eta = -1$ for v.

We know that there is no gradient for which $\eta = -1$, which Ω annihilates. Consequently, for the case $\eta = -1$,

$$\left\{1 - \frac{O\Omega}{1^2} + \frac{O^2\Omega^2}{1^2.\,2^2} - \frac{O^3\Omega^3}{1^2.\,2^2.\,3^2} + \dots\right\} v = 0,$$

a result which will presently be proved to hold when, for v, η is any negative integer whatever.

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4. It is now proposed to show that the identity which expresses u in the form Ωv, viz.,

$$\left\{1 - \frac{1}{1.\eta}\,\Omega O + \frac{1}{1.2.\eta.\eta+1}\,\Omega O^2\Omega - \frac{1}{1.2.3.\eta.\eta+1.\eta+2}\,\Omega O^3\Omega^2 + \dots\right\} u$$
$$= 0 \quad \dots\dots\dots\dots(1),$$

may be more elegantly written

$$\left\{1 - \frac{\Omega O}{1^2} + \frac{\Omega^2 O^2}{1^2.2^2} - \frac{\Omega^3 O^3}{1^2.2^2.3^2} + \dots\right\} u = 0 \quad \dots\dots\dots(2),$$

in which the characteristic η is not explicitly present, though it is required to be a positive number.

This is equally a fact whether we are dealing with a single set of constituents $a_0, a_1, a_2, \dots a_n$, as in my former paper, so that η denotes the excess $in - 2w$, and Ω, O are as in (1) and (2) of § 2, but without the Σ symbols of summation; or with a system of sets, as in § 2.

To aid us in the performance of the transformation, we have a formula of Hilbert's, viz.,

$$O^r\Omega^s = \Omega^s O^r - (\eta - r + s)\, rs\Omega^{s-1} O^{r-1}$$
$$+ \frac{(\eta - r + s)(\eta - r + s + 1)}{1.2}\, r(r-1)\, s(s-1)\, \Omega^{s-2} O^{r-2} - \dots \quad \dots(3),$$

where the operation is on any gradient whose excess $\Sigma(in) - 2w$ is equal to η. For the case of $s = 1$ this is the well known

$$O^r\Omega = \Omega O^r - r(\eta - r + 1)\, O^{r-1};$$

and for higher values of s it is established by mathematical induction.

For the present purpose, we have to put $r-1$ for s in (3), and

operate with Ω, thus getting

$$\Omega O^r \Omega^{r-1} = \Omega^r O^r - (\eta-1)\,\eta r\,(r-1)\,\Omega^{r-1} O^{r-1}$$
$$+\frac{(\eta-1)\,\eta}{1.2}\,r\,(r-1)^2\,(r-2)\,\Omega^{r-2} O^{r-2}$$
$$-\frac{(\eta-1)\,\eta\,(\eta+1)}{1.2.3}\,r\,(r-1)^2\,(r-2)^2\,(r-3)\,\Omega^{r-3} O^{r-3}+\dots,$$

and then to give r successively the values 1, 2, 3, ..., and substitute in (1).

In this way the coefficient of $\Omega^r O^r$, in what the left-hand side of (1) becomes, is seen to be

$$\frac{(-1)^r}{r!}\left\{\frac{1}{\eta.\eta+1\dots\eta+r-1}+r\,\frac{\eta-1}{\eta.\eta+1\dots\eta+r}\right.$$
$$+\frac{r.r+1}{1.2}\cdot\frac{\eta-1}{\eta+1.\eta+2\dots\eta+r+1}$$
$$\left.+\frac{r.r+1.r+2}{1.2.3}\cdot\frac{\eta-1}{\eta+2.\eta+3\dots\eta+r+2}+\dots\right\},$$

which

$$=\frac{(-1)^r(\eta-1)}{(r!)^2}\left\{\frac{(\eta-2)!\,r!}{(\eta+r-1)!}+r\,\frac{(\eta-1)!\,r!}{(\eta+r)!}+\frac{r.r+1}{1.2}\,\frac{\eta!\,r!}{(\eta+r+1)!}+\dots\right\}$$
$$=\frac{(-1)^r(\eta-1)}{(r!)^2}\int_0^1(1-x)^r\left(x^{\eta-2}+rx^{\eta-1}+\frac{r.r+1}{1.2}\,x^{\eta}+\dots\right)dx$$
$$=\frac{(-1)^r(\eta-1)}{(r!)^2}\int_0^1 x^{\eta-2}\,dx$$
$$=\frac{(-1)^r}{(r!)^2}.$$

Thus (1) becomes (2), as stated.

For the special case of $\eta=1$, the last part of this work is both inapplicable and unnecessary. The conclusion is the same as in general.

The simplest form for a gradient which, when operated on by Ω, produces u, a given gradient for which η is a positive number, is

$$\Omega^{-1}u=\left\{\frac{O}{1^2}-\frac{\Omega O^2}{1^2.2^2}+\frac{\Omega^2 O^3}{1^2.2^2.3^2}-\dots\right\}u \quad \dots\dots\dots\dots\dots(3),$$

to which, of course, may be added, as (so to speak) an arbitrary constant, any seminvariant whose type is that of Ou.

In any general property of a gradient, we may, of course, interchange Ω, O, w, η with O, Ω, $\Sigma\,(in)-w$, $-\eta$, respectively, thus merely interchanging the first and last, second and last but one, &c., constituents of each set. Companion then to the equivalent facts (1) and (2) of the present article, we have a pair of equivalent facts with regard to a product or linear function u of products of any the same type w, i, i', i'', ... for which η or $\Sigma\,(in)-2w$ is negative, equal to $-\eta'$, say, viz.,

$$\left\{1-\frac{1}{1.\eta'}O\Omega+\frac{1}{1.2.\eta'.\eta'+1}O\Omega^2O-\frac{1}{1.2.3.\eta'.\eta'+1.\eta'+2}O\Omega^3O^2+\dots\right\}u$$
$$=0 \quad \dots\dots\dots\dots\dots\dots\dots\dots(4),$$

$$\left\{1-\frac{O\Omega}{1^2}+\frac{O^2\Omega^2}{1^2.2^2}-\frac{O^3\Omega^3}{1^2.2^2.3^2}+\dots\right\}u=0 \quad \dots\dots\dots\dots(5).$$

To the intermediate case of $\eta=0$, neither (1) and (2) nor (4) and (5) apply. It will be seen later what the operators on the left-hand sides produce from a gradient u of this type.

5. Other forms, of some interest, to which (1) or (2) of the preceding article may be reduced, are obtained by noticing that

$$\begin{aligned}\Omega^rO^r &= \Omega^{r-1}\left\{O^r\Omega+r\,(\eta-r+1)\,O^{r-1}\right\}\\ &= \Omega^{r-1}O^{r-1}\left\{O\Omega+r\,(\eta-r+1)\right\}\\ &= \Omega^{r-2}O^{r-2}\left\{O\Omega+(r-1)(\eta-r+2)\right\}\left\{O\Omega+r\,(\eta-r+1)\right\}\\ &= \dots\\ &= (O\Omega+1.\eta)(O\Omega+2.\eta-1)(O\Omega+3.\eta-2)\dots(O\Omega+r.\eta-r+1)\end{aligned}$$
$$\dots\dots\dots\dots\dots\dots(1).$$

We thus see that § 4 (1) may be written

$$\left\{1-\frac{\frac{O\Omega}{1}+\eta}{1}+\frac{\left(\frac{O\Omega}{1}+\eta\right)\left(\frac{O\Omega}{2}+\eta-1\right)}{1.2}\right.$$
$$\left.+\frac{\left(\frac{O\Omega}{1}+\eta\right)\left(\frac{O\Omega}{2}+\eta-1\right)\left(\frac{O\Omega}{3}+\eta-2\right)}{1.2.3}+\dots\right\}u=0$$
$$\dots\dots\dots\dots(2).$$

Thus, if a theory exists of the function of z, n,

$$1-\frac{\frac{z}{1}+n}{1}+\frac{\left(\frac{z}{1}+n\right)\left(\frac{z}{2}+n-1\right)}{1.2}$$

$$-\frac{\left(\frac{z}{1}+n\right)\left(\frac{z}{2}+n-1\right)\left(\frac{z}{3}+n-2\right)}{1.2.3}+\dots \quad \dots (3),$$

it will have its bearing on the present subject. (*Cf.* § 14 below.)

Another form of (1) is

$$\Omega^r O^r = \Omega O\,(\Omega O+\eta-2)(\Omega O+2.\eta-3)\dots(\Omega O+\overline{r-1}.\eta-r)\dots(4)$$

so that we have also, η being a positive number for u,

$$\left\{1-\frac{\Omega O}{1^2}+\frac{\Omega O(\Omega O+\eta-2)}{1^2.2^2}-\frac{\Omega O\,(\Omega O+\eta-2)(\Omega O+2.\eta-3)}{1^2.2^2.3^2}+\dots\right\}u$$

$$=0 \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(5).$$

We have also, as companion to (1) and (4),

$$O^r\Omega^r = (\Omega O-1.\eta)(\Omega O-2.\eta+1)(\Omega O-3.\eta+2)\dots(\Omega O-r.\eta+r-1)$$

$$= O\Omega\,(O\Omega-1.\eta+2)(O\Omega-2.\eta+3)\dots(O\Omega-\overline{r-1}.\eta+r) \quad \dots\dots(6),$$

so that forms, resembling (2) and (5) above, of § 4 (4) and (5), which apply to gradients u for which the integer η is negative, are at once written down.

Such transformations might be multiplied. Very useful facts for such purposes are that

$$\Omega^r O^r.\Omega^s O^s = \Omega^s O^s.\Omega^r O^r,$$

$$O^r\Omega^r.O^s\Omega^s = O^s\Omega^s.O^r\Omega^r,$$

$$\Omega^r O^r.O^s\Omega^s = O^s\Omega^s.\Omega^r O^r,$$

which are at once clear from the factorized forms (1), (4), (6), since factors like $\Omega O+p$, $\Omega O+q$ are commutative with one another.

6. The explicit absence of η in (2) and (5) of § 4 is surprising, seeing that a limitation of its range of values is in each case implied. Moreover the proof of § 4 is only convincing upon close attention.

An independent demonstration will therefore be given, and this will have the effect of determining the effect of the operator

$$Z_1 \equiv 1 - \frac{\Omega O}{1^2} + \frac{\Omega^2 O^2}{1^2.2^2} - \frac{\Omega^3 O^3}{1^2.2^2.3^2} + \dots \qquad \dots\dots\dots\dots(1)$$

on gradients for which η is not positive, as well as on others, and of the operator

$$Z_2 \equiv 1 - \frac{O\Omega}{1^2} + \frac{O^2\Omega^2}{1^2.2^2} - \frac{O^3\Omega^3}{1^2.2^2.3^2} + \dots \qquad \dots\dots\dots\dots(2)$$

on gradients for which η is not, as well as those for which it is, negative.

A number of extensive and widely distinct classes of cases may first be mentioned, in which the verification is easy.

(i.) When $\eta = -1$, it has been proved, from Hilbert's theorem in § 3, that $Z_2 u = 0$. Consequently, when $\eta = +1$, we have $Z_1 u = 0$.

(ii.) Let u be a single constituent a_w of the set $a_0, a_1, \dots a_n$. In this case

$$Z_1 a_w = \left\{ 1 - \frac{w+1}{1} \cdot \frac{n-w}{1} + \frac{w+1 \,.\, w+2}{1.2} \cdot \frac{n-w \,.\, n-w-1}{1.2} - \dots \right\} a_w,$$

which is readily proved to vanish when $n > 2w$.

(iii.) Let u be a seminvariant. Then

$$Z_1 u = \left\{ 1 - \frac{\eta}{1} + \frac{\eta \,.\, \eta-1}{1.2} - \frac{\eta \,.\, \eta-1 \,.\, \eta-2}{1.2.3} + \dots \right\} u,$$

which vanishes, since η is a positive integer.

(iv.) Another easily tested case is that when $n = 1$.

7. To examine the effect of Z_1 on any gradient, we notice that $Z_1 u$ is the term independent of t in the expansion, in positive and negative integral powers of t, of

$$e^{-t^{-1}\Omega} e^{tO} u.$$

This expansion, it is to be noticed, terminates both ways. For some power of O annihilates the rational integral function u, and some power of Ω annihilates any term which does not vanish in $e^{tO} u$. Thus, if we can obtain another terminating expression for $e^{-t^{-1}\Omega} e^{tO} u$, the two must be identical.

Take first a single constituent a_r of the set $a_0, a_1, \ldots a_n$. We have

$$e^{t\Omega} a_r = a_r + (n-r)\, a_{r+1} t + \frac{(n-r)(n-r-1)}{1.2} a_{r+2} t^2 + \ldots + a_n t^{n-r}$$

$$= \frac{(n-r)!}{n!} \frac{d^r}{dt^r} \left\{ a_0 + n a_1 t + \frac{n(n-1)}{1.2} t^2 + \ldots + a_n t^n \right\}.$$

Now operate on this with $e^{\tau\Omega}$, taking τ for the present independent of t, so that $e^{\tau\Omega}$ and $\frac{d^r}{dt^r}$ are commutative. We get

$$e^{\tau\Omega} e^{t\Omega} a_r = \frac{(n-r)!}{n!} \frac{d^r}{dt^r} \left\{ a_0 + n(a_1 + a_0 \tau)\, t \right.$$

$$+ \frac{n(n-1)}{1.2} (a_2 + 2a_1\tau + a_0\tau^2)\, t^2 + \ldots$$

$$\left. \ldots + (a_n + n a_{n-1}\tau + \ldots + a_0 \tau^n)\, t^n \right\}$$

$$= \frac{(n-r)!}{n!} \frac{d^r}{dt^r} \left\{ a_0 (1+\tau t)^n + n a_1 t\, (1+\tau t)^{n-1} \right.$$

$$\left. + \frac{n(n-1)}{1.2} a_2 t^2 (1+\tau t)^{n-2} + \ldots + a_n t^n \right\}.$$

In this, after the differentiations with regard to t are performed, we are going to give to $1+\tau t$ the value zero, *i.e.*, to make $\tau = -t^{-1}$.

Now, if $s > r$,

$$\left(\frac{d}{dt}\right)^r \{ t^{n-s} (1+\tau t)^s \}$$

has $(1+\tau t)^{s-r}$ for a factor throughout, and so vanishes when

$$1 + \tau t = 0.$$

Also, by application of Leibnitz's theorem, if $s =$ or $< r$, and is a positive integer, the only part of $\left(\frac{d}{dt}\right)^r \{ t^{n-s} (1+\tau t)^s \}$ which does not vanish when $1 + \tau t = 0$ is

$$\frac{r!}{s!\,(r-s)!} \cdot \frac{d^{r-s}}{dt^{r-s}} (t^{n-s}) \cdot \frac{d^s}{dt^s} \{(1+\tau t)^s\},$$

i.e., is

$$\frac{r!}{s!\,(r-s)!} \cdot \frac{(n-s)!}{(n-r)!} t^{n-r} \cdot s!\, \tau^s,$$

which

$$= \frac{(n-s)!\; r!}{(n-r)!\; (r-s)!} t^{n-r} \tau^s.$$

Consequently, the part of $e^{\tau\Omega}e^{t\Omega}a_r$ which does not vanish when we put $1+\tau t = 0$, is

$$t^{n-r}\frac{r!}{n!}\left\{\frac{n!}{r!\,(n-r)!}\cdot\frac{(n-r)!}{1}a_{n-r}\tau^{r}\right.$$
$$+\frac{n!}{(r-1)!\,(n-r+1)!}\cdot\frac{(n-r+1)!}{1!}a_{n-r+1}\tau^{r-1}$$
$$\left.+\frac{n!}{(r-2)!\,(n-r+2)!}\cdot\frac{(n-r+2)!}{2!}a_{n-r+2}\tau^{r-2}+\dots+1\,\frac{n!}{r!}a_n\right\},$$

i.e. $$t^{n-r}\left\{a_{n-r}\tau^{r}+ra_{n-r+1}\tau^{r-1}+\frac{r(r-1)}{1.2}a_{n-r+2}\tau^{r-2}+\dots+a_n\right\};$$

so that, upon putting $1+\tau t = 0$,

$$e^{-t^{-1}\Omega}e^{t\Omega}a_r$$
$$=(-1)^r t^{n-2r}\left\{a_{n-r}-ra_{n-r+1}t+\frac{r(r-1)}{1.2}a_{n-r+2}t^2-\dots+(-1)^r a_n t^r\right\}$$
$$=(-1)^r t^{n-2r}e^{-t\Omega}a_{n-r}\;\dots\dots\dots\dots(1).$$

In verification, it may be noticed that this gives, upon operating on both sides with $e^{t^{-1}\Omega}$,

$$e^{t^{-1}\Omega}.e^{-t\Omega}a_{n-r}=(-1)^r t^{2r-n}e^{t\Omega}a_r$$
$$=(-1)^{n-r}(-t)^{n-2(n-r)}e^{t\Omega}a_r\;\dots\dots\dots\dots(2),$$

which is correctly the result of interchanging t and $-t$, r and $n-r$ in (1).

Now $e^{-t\Omega}(uv)=e^{-t\Omega}u\,.\,e^{-t\Omega}v$, by a well known property of linear differential operators. Also

$$e^{-t^{-1}\Omega}e^{t\Omega}(uv)=e^{-t^{-1}\Omega}(e^{t\Omega}u\,.\,e^{t\Omega}v)$$
$$=e^{-t^{-1}\Omega}e^{t\Omega}u\,.\,e^{-t^{-1}\Omega}e^{t\Omega}v.$$

Accordingly, we can pass from a single a_r to any product of integral powers of constituents, chosen from the same set $a_0, a_1, \dots a_n$, or from different sets. Thus, in the case of one set, $\lambda_0, \lambda_1, \dots \lambda_n$ being positive integers,

$$e^{-t^{-1}\Omega}e^{t\Omega}.\,a_0^{\lambda_0}a_1^{\lambda_1}\dots a_n^{\lambda_n}=(-1)^w t^{in-2w}e^{-t\Omega}.\,a_n^{\lambda_0}a_{n-1}^{\lambda_1}\dots a_0^{\lambda_n}\dots\dots(3),$$

and in the case of different sets the like fact holds, viz., that $e^{-t^{-1}\Omega}e^{tO}$, operating on a product, is equal to $(-1)^w t^\eta$ times e^{-tO} operating on the conjugate product obtained by putting the last, last but one, &c. constituents of every set for the first, second, &c. of that set, where

$$\eta = \Sigma\,(in) - 2w.$$

8. Now in the expansion of $t^\eta e^{-tO}$ there is not, or is, a term free from t, according as η, an integer or zero, is greater or not greater than zero. Consequently, for the case of one set, we have the conclusions

(i.) if $in - 2w > 0$, $Z_1 u = 0$;

(ii.) if $in - 2w \not> 0$,

$$Z_1 . a_0^{\lambda_0} a_1^{\lambda_1} \dots a_n^{\lambda_n} = (-1)^w \frac{(-1)^{2w-in}}{(2w-in)!} O^{2w-in} a_n^{\lambda_0} a_{n-1}^{\lambda_1} \dots a_0^{\lambda_n}$$

$$= (-1)^{in-w} \frac{1}{(2w-in)!} O^{2w-in} a_n^{\lambda_0} a_{n-1}^{\lambda_1} \dots a_0^{\lambda_n};$$

of which latter one case may well be written separately, viz.,

(iii.) if $in - 2w = 0$,

$$Z_1 . a_0^{\lambda_0} a_1^{\lambda_1} \dots a_n^{\lambda_n} = (-1)^w a_n^{\lambda_0} a_{n-1}^{\lambda_1} \dots a_n^{\lambda_n}.$$

For the case of products of several sets of constituents, we have at once the like conclusions, which need not be written down at length.

We may now, of course, take, instead of one product of positive integral powers of the constituents, a gradient or linear function of such products of the same type w, i, i', i'', ..., and may summarize by saying that a gradient u whose η characteristic $\Sigma\,(in) - 2w$ is positive has Z_1 for an annihilator; while if the η of u is not positive and different from zero the effect of operating on u with Z_1 is to interchange in u the first and last, second and last but one, &c. constituents of every set, to operate on the result with $O^{-\eta}$, and to apply the factor $(-1)^{w+\eta} \dfrac{1}{(-\eta)!}$.

It is perhaps hardly necessary to state at length the strictly companion facts with regard to the second operator Z_2 of § 6. We have merely to interchange w and $\Sigma\,(in) - w$, η and $-\eta$, in the facts with regard to Z_1.

9. A study of the operator Z_1 is not complete without some reference to its effect on functions of the constituents which have the properties of homogeneity in every set of constituents, and isobarism on the whole, but which differ from gradients in not being rational and integral. It is without difficulty proved that the equality (3) of § 7 holds with regard to products in which $\lambda_0, \lambda_1, \dots \lambda_n$ are not positive integers. The same cannot, however, be said without further investigation of the conclusions (i.), (ii.), (iii.) of the same article. Though the two sides of § 7 (3) are equal, their expansions formed by ordinary methods will in the case now contemplated extend to infinity—on the left as a rule in both directions—and, as we have no general information as to convergency, the identity, term for term, of the two expansions is not established.

Still we cannot doubt that $Z_1 u = 0$ holds in many cases when u is not rational and integral. What has been proved in § 4 is more than there stated. It would seem to be that whatever u be, provided its η be the same throughout, and *not a negative integer or zero*, the operators

$$1-\frac{1}{1.\eta}\Omega O+\frac{1}{1.2.\eta.\eta+1}\Omega O^2\Omega-\frac{1}{1.2.3.\eta.\eta+1.\eta+2}\Omega O^3\Omega^2+\dots \qquad \text{.................(1)}$$

and

$$1-\frac{\Omega O}{1^2}+\frac{\Omega^2 O^2}{1^2.2^2}-\frac{\Omega^3 O^3}{1^2.2^2.3^2}+\dots \qquad \text{..................(2),}$$

which latter we call Z_1, derive from u expressions which are identical if convergent. Now when it was proved (§ 3 above, and Vol. XXIII., p. 300) that the operator (1) annihilates gradients of positive η, it was at the same time disproved for functions u which are not either such gradients or else functions which, while, it may be, irrational or fractional, have $O^m\Omega^m$ as an annihilator for some value or other of the number m. Thus, besides gradients of positive η, Z_1 annihilates such functions.* There is besides the reservation of cases when η is zero or a negative integer, for which Z_1 is not in general identical with (1).

The difficulties of a complete study of the effect of Z_1 on products, or linear functions of products, which are not rational and integral,

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* A gradient multiplied by a negative power of a_0 is such a function.

appear to be very great, in general. In the first case, when $n = 1$, *i.e.* when there are only two constituents a_0, a_1, or a, b, there is, however, no such difficulty. A full investigation of that case occupies the remainder of this paper. Though the arithmetical method used is only one of special application, the results may be of use in suggesting probabilities for higher values of n.

10. Take $n = 1$, and a single set a, b of constituents only. For this case $\Omega = a\dfrac{d}{db}$ and $O = b\dfrac{d}{da}$. We have then

$$Z_1 . a^\lambda b^\mu = \left\{1 - \frac{\lambda . \mu+1}{1^2} + \frac{\lambda . \lambda-1 . \mu+1 . \mu+2}{1^2 . 2^2} \right.$$

$$\left. - \frac{\lambda . \lambda-1 . \lambda-2 . \mu+1 . \mu+2 . \mu+3}{1^2 . 2^2 . 3^2} + \dots \right\} a^\lambda b^\mu$$

$$= \left\{1 + \frac{-\lambda(\mu+1)}{1^2} + \frac{-\lambda(-\lambda+1)(\mu+1)(\mu+2)}{1^2 . 2^2} + \dots \right\} a^\lambda b^\mu$$

$$= F(-\lambda, \mu+1, 1, 1)\, a^\lambda b^\mu;$$

where $F(\alpha, \beta, \gamma, x)$ is the ordinary notation for a hypergeometric series.

Now the hypergeometric series $F(\alpha, \beta, \gamma, 1)$ terminates if either α or β is a negative integer or zero, and is convergent if $\gamma - \alpha - \beta$ is positive. The special case of γ being zero or a negative integer does not here arise. We have then that $Z_1 . a^\lambda b^\mu$ is an intelligible arithmetical multiple of $a^\lambda b^\mu$,

(α) if λ is a positive integer or zero,

(β) if μ is a negative integer, not including zero,

(γ) if, even though neither (α) nor (β) is the case, $\lambda - \mu > 0$, *i.e.*, $in - 2w > 0$.

On the contrary, when neither of these conditions holds, $Z_1 u$ is infinite or unintelligible.

Now (Forsyth's *Differential Equations*, § 126),

$$F(-\lambda, \mu+1, 1, 1) = \frac{\Gamma(1)\,\Gamma(\lambda-\mu)}{\Gamma(\lambda+1)\,\Gamma(-\mu)}.$$

Thus, taking the above three cases of intelligibility of $Z_1 . a^\lambda b^\mu$,

(i.) if λ is a positive integer or zero, whatever μ be,

$$Z_1 . a^\lambda b^\mu = \frac{(\lambda-\mu-1)(\lambda-\mu-2)\dots(-\mu)}{\lambda!} a^\lambda b^\mu;$$

(ii.) if μ is a negative integer, whatever λ be,

$$Z_1 . a^\lambda b^\mu = \frac{(\lambda-\mu-1)(\lambda-\mu-2)\dots(\lambda+1)}{(-\mu-1)!} a^\lambda b^\mu;$$

(iii.) if $\lambda-\mu > 0$,

$$Z_1 . a^\lambda b^\mu = \frac{\Gamma(\lambda-\mu)}{\Gamma(\lambda+1)\,\Gamma(-\mu)} a^\lambda b^\mu.$$

In particular, we may at once see what laws λ and μ must satisfy, that $Z_1 . a^\lambda b^\mu$ may vanish. On examination the laws given by (i.) and (ii.) for the purpose are included in the wider law given by (iii.). And this law is afforded by the condition that $\Gamma(\lambda+1)\,\Gamma(-\mu)$ be infinite, which is the case only if either λ be a negative integer or μ a positive integer or zero.

Thus, for the case of $n = 1$, the accurate extension of the theorem that $Z_1 . a^\lambda b^\mu = 0$, when λ and μ are positive integers such that

$$\lambda-\mu = in-2w \not< 0,$$

is

"$Z_1 . a^\lambda b^\mu = 0$, when and only when $\lambda-\mu > 0$, and, either μ is a positive integer or zero, or λ a negative integer."

It will be noticed that, in accordance with § 9, a number m can be chosen sufficiently great that $O^m \Omega^m$ annihilates $a^\lambda b^\mu$.

11. It will be well further to examine whether what (ii.) and the included (iii.) of § 8 become, for the case $n = 1$, hold or have intelligible representatives for any wider values of λ and μ than those for which they have been proved. With $n = 1$, § 8 (2) may be stated: "If λ and μ are positive integers, zero included, such that $\lambda \not> \mu$,

$$Z_1 . a^\lambda b^\mu = (-1)^\lambda \frac{1}{(\mu-\lambda)!} \left(b\frac{d}{da}\right)^{\mu-\lambda} b^\lambda a^\mu$$

$$= (-1)^\lambda \frac{\mu(\mu-1)\dots(\lambda+1)}{(\mu-\lambda)!} a^\lambda b^\mu$$

$$= (-1)^\lambda \frac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda+1)\,\Gamma(\lambda+1)} a^\lambda b^\mu \text{"} \quad \dots\dots\dots(1).$$

Now the cases in which $\lambda-\mu \not> 0$, for which $Z_1.a^\lambda b^\mu$ is finite, must all come under (i.) or (ii.) or both of § 10.

The former tells us that, when $\lambda-\mu \not> 0$, and λ is a positive integer or zero, whatever not smaller quantity μ be,

$$Z_1.a^\lambda b^\mu = (-1)^\lambda \frac{\mu(\mu-1)(\mu-2)\dots(\mu-\lambda+1)}{\lambda!} a^\lambda b^\mu$$

$$= (-1)^\lambda \frac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda+1)\,\Gamma(\lambda+1)} a^\lambda b^\mu;$$

so that (1) holds for this generalized law of λ and μ.

And again, the latter tells us that, when μ is a negative integer, and λ any quantity such that $\lambda-\mu \not> 0$,

$$Z_1.a^\lambda b^\mu = (-1)^{\mu+1} \frac{(-\lambda-1)(-\lambda-2)\dots(-\lambda+\mu+1)}{(-\mu-1)!} a^\lambda b^\mu$$

$$= (-1)^{\mu+1} \frac{\Gamma(-\lambda)}{\Gamma(\mu-\lambda+1)\,\Gamma(-\mu)} a^\lambda b^\mu.$$

12. Collecting results as to the case $n = 1$, we have that, in the equality $Z_1.a^\lambda b^\mu = \zeta_1(\lambda, \mu)\, a^\lambda b^\mu$,

(A) if $\lambda > \mu$ (*i.e.* $\eta > 0$),

$$\zeta_1(\lambda, \mu) = \frac{\Gamma(\lambda-\mu)}{\Gamma(\lambda+1)\,\Gamma(-\mu)},$$

and is not infinite. In particular, $\zeta_1(\lambda, \mu) = 0$, when either μ is a positive integer or zero, whatever greater quantity λ may be, or when λ is a negative integer, whatever algebraically less quantity μ be;

(B) if $\lambda \not> \mu$ (*i.e.* $\eta \not> 0$), $\zeta_1(\lambda, \mu)$ never vanishes, but is generally a divergent series. It has, however, a finite expression in two cases, (1) if λ is a positive integer or zero, in which case its value is $(-1)^\lambda \dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\lambda+1)\,\Gamma(\lambda+1)}$, and (2) if μ is a negative integer, in which case its value is $(-1)^{\mu+1} \dfrac{\Gamma(-\lambda)}{\Gamma(\mu-\lambda+1)\,\Gamma(-\mu)}$. The case of λ and μ both positive integers is included in (1).

The companion facts with regard to $Z_2.a^\lambda b^\mu$ can be readily deduced.

13. The identity of the expressions produced, when both terminate or converge, by operations on a given product with

$$H_1 = 1 - \frac{1}{1.\eta}\,\Omega O + \frac{1}{1.2.\eta.\eta+1}\,\Omega O^2\Omega - \ldots,$$

and
$$Z_1 = 1 - \frac{\Omega O}{1^2} + \frac{\Omega^2 O^2}{1^2.2^2} - \ldots,$$

is well exhibited in the present case of $n = 1$.

If, in fact, we write $H_1.a^\lambda b^\mu = \eta_1(\lambda, \mu)\, a^\lambda b^\mu$, we see at once that, since now $\eta = \lambda - \mu$,

$$\eta_1(\lambda, \mu) = 1 - \frac{\lambda.\mu+1}{1.\lambda-\mu} + \frac{\lambda.\lambda+1.\mu+1.\mu}{1.2.\lambda-\mu.\lambda-\mu+1}$$

$$- \frac{\lambda.\lambda+1.\lambda+2.\mu+1.\mu.\mu-1}{1.2.3.\lambda-\mu.\lambda-\mu+1.\lambda-\mu+2} + \ldots$$

$$= F_1(\lambda, -\mu-1, \lambda-\mu, 1)$$

$$= \frac{\Gamma(\lambda-\mu)\,\Gamma(1)}{\Gamma(-\mu)\,\Gamma(\lambda+1)},$$

which is also the value of $\zeta_1(\lambda, \mu)$ obtained in § 10.

But $\eta_1(\lambda, \mu)$ is never divergent, though it is infinite if $\lambda - \mu$ be zero or a negative integer, for here

$$\gamma - \alpha - \beta = \lambda - \mu - (\lambda - \mu - 1) = 1,$$

and is positive. We saw, however, that for $\zeta_1(\lambda, \mu)$ to be convergent it is necessary that $\lambda - \mu$ be positive, the other cases of its finiteness being those when either λ is a positive integer or zero, or μ a negative integer. Thus $H_1.a^\lambda b^\mu$ and $Z_1.a^\lambda b^\mu$ are identical when neither is infinite, but the cases of their finiteness are not coextensive, the former having the advantage.

This identity of H_1 and Z_1 is then, for $n = 1$, an application of Gauss's equality of two hypergeometric series,

$$F(\alpha, \beta, \gamma, 1) = F(-\alpha, -\beta, \gamma-\alpha-\beta, 1).$$

The two are equal when both are convergent or finite; but the conditions of their convergency are respectively that $\gamma - \alpha - \beta$ and γ be positive.

It will be of great interest if the study of H_1 and Z_1 for higher values of n lead hereafter to conclusions which are generalizations of this arithmetical theorem.

14. We are also led to an interesting arithmetical conclusion by considering, for the case of $n = 1$, the identity with H_1 or Z_1 of the operator (2) of § 5. We are thus given, in fact, an expression for the sum of a series of the form (3) of § 5.

Since, in the present case,

$$O\Omega . a^\lambda b^\mu = b\frac{d}{da}\left(a\frac{d}{db}\right) a^\lambda b^\mu = (\lambda+1)\,\mu . a^\lambda b^\mu,$$

the effect of the operator now under consideration is to produce, from $a^\lambda b^\mu$,

$$\left\{1-\frac{\frac{\mu(\lambda+1)}{1}+\lambda-\mu}{1}+\frac{\left(\frac{\mu(\lambda+1)}{1}+\lambda-\mu\right)\left(\frac{\mu(\lambda+1)}{2}+\lambda-\mu\right)}{1.2}-\ldots\right\} a^\lambda b^\mu,$$

which must be identical with $\zeta_1(\lambda, \mu)\, a^\lambda b^\mu$ or $\eta_1(\lambda, \mu)\, a^\lambda b^\mu$.

We have, consequently, upon putting z, η for $\mu(\lambda+1)$, $\lambda-\mu$,

$$1-\frac{\frac{z}{1}+\eta}{1}+\frac{\left(\frac{z}{1}+\eta\right)\left(\frac{z}{2}+\eta-1\right)}{1.2}-\frac{\left(\frac{z}{1}+\eta\right)\left(\frac{z}{2}+\eta-1\right)\left(\frac{z}{3}+\eta-2\right)}{1.2.3}+\ldots$$

$$= F\left\{-\tfrac{1}{2}\left[\eta-1+\sqrt{(\eta+1)^2+4z}\right],\ -\tfrac{1}{2}\left[\eta-1-\sqrt{(\eta+1)^2+4z}\right], 1, 1\right\}$$

$$= F\left\{\tfrac{1}{2}\left[\eta-1+\sqrt{(\eta+1)^2+4z}\right],\ \tfrac{1}{2}\left[\eta-1-\sqrt{(\eta+1)^2+4z}\right], \eta, 1\right\}$$

$$= \frac{\Gamma(\eta)\,\Gamma(1)}{\Gamma\left\{\tfrac{1}{2}\left[\eta+1+\sqrt{(\eta+1)^2+4z}\right]\right\}\Gamma\left\{\tfrac{1}{2}\left[\eta+1-\sqrt{(\eta+1)^2+4z}\right]\right\}}.$$

In like manner, by aid of the operator (5) of § 5, we obtain, for the series

$$1-\frac{z}{1^2}+\frac{z(z+\eta-2)}{1^2.2^2}-\frac{z(z+\eta-2)(z+2\overline{\eta-3})}{1^2.2^2.3^2}+\ldots,$$

the expression

$$\frac{\Gamma(\eta)\,\Gamma(1)}{\Gamma\left\{\tfrac{1}{2}\left[\eta+1+\sqrt{(\eta-1)^2+4z}\right]\right\}\Gamma\left\{\tfrac{1}{2}\left[\eta+1-\sqrt{(\eta-1)^2+4z}\right]\right\}}.$$

On certain General Limitations affecting Hyper-magic Squares. *By* SAMUEL ROBERTS. Received and read November 10th, 1892.

1. This paper does not aim at making any addition to the known ways of constructing magic squares.*

Hyper-magic squares, as I regard them, include those called by the late M. E. Lucas† "carrés diaboliques," and also treated of by Rev. A. H. Frost, under the designation "nasik squares."‡ The special form is of ancient origin. The second method given in the fragment by Moschopulus (probably of the fourteenth century) is a general one for forming such squares, and they have been discussed by various modern authors. My object is to show some limitations to which they are subject when the elements are positive or negative integers. Incidentally it will appear that hyper-magic squares of oddly even orders cannot be formed of series of consecutive natural numbers.§ There is some reason to believe that much ingenuity has

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* Notwithstanding this remark, it has been imagined that I contemplated the actual construction of hyper-magic squares having consecutive natural numbers as elements. So far is this from being the case, I have not, unless inadvertently, shown that such squares exist. It was not necessary, since my conclusions are of a negative kind.

† The subject has been brought into connexion with the "Geometry of Tissues," by M. Lucas, and others (*Principii fondamentali della Geometria dei Tessuti*, per Edoardo Lucas, Torino, 1880; see also *Récréations Mathématiques*, par M. E. Lucas, Introduction, t. I., p. XVIII.).

‡ I do not say that hyper-magic squares include nasik squares, but that they include "carrés diaboliques," which, I take it, are hyper-magic squares made up of natural numbers from 1 to n^2 (v. § 2).

The first definition of nasik squares (*Quarterly Journal of Mathematics*, VII., pp. 93, 94) apparently makes "carrés diaboliques" coextensive with them.

A later definition (*Quar. Jour.*, XV., p. 34) is in the following terms: "A square containing n cells on each side, in which are placed the natural numbers from 1 to n^2, in such an order that the constant sum $\frac{1}{2}n(n^2+1)$ is obtained by adding the numbers on n of the cells, those n cells lying in a variety of different directions, and their relative position in each direction being defined by simple laws."

I should not presume to limit the comprehensiveness of this definition.

§ With regard to this, I have been referred to the following passage in Rev. A. H. Frost's paper (*Quarterly Journal of Mathematics*, XV., p. 49):—

"Nasik Squares of the form $2(2n+1)$ cannot be filled with consecutive natural numbers from 1 to $4(2n+1)^2$ either by this or the process adopted in the previous paper; for it will be found that, as in the squares of the form $4n$, we have to give (*e.g.*, the case of 6^2) $p_1, p_2, \ldots p_6$ such values that the sum of 3 equals the sum of the

been fruitlessly employed in trying to form such squares. It may be well to mention here that a very interesting historical essay on the subject of magic squares has been published by Dr. Siegmund Günther, in his work entitled *Vermischte Untersuchungen zur Geschichte der Mathematischen Wissenschaften*, Leipsic, 1876. This work contains the fragment of Moschopulus. The short historical notices found in ordinary books of reference are necessarily very inadequate.

2. Consider the square array

$$\left.\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & a_{14} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \ldots\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & \ldots\ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \ldots\ldots & a_{nn} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (A).$$

There are n rows, n columns, and two principal diagonals. We may, however, reckon $2(n-1)$ secondary or broken diagonals, each being made up of a series of literal elements parallel to a principal diagonal together with a complementary series parallel to the same diagonal, but on the other side of it, the two series being composed of n elements. The claim of these pairs of series to be regarded as diagonals is apparent if we suppose the arrangement to be applied by deformation to the surface of an anchor ring. The broken diagonals then become complete.

Further, let the numerical values of the literal elements be such that the sum of each row, of each column, and of each diagonal, is the same. The square is then hyper-magic. The number n is the order, and the common sum-value of the rows, columns, and diagonals is the weight of the square (A).*

3. It is necessary to take the cases of odd and even orders separately.

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other 3; but, as the sum of 6 = $\frac{1}{2}\,6\cdot 7$, an odd number, they cannot be thus divided; but if we pass over one of the p's and r's, making the r's, 1, 2, 3, 4, 5, 7, and the p's = 0, 1, 2, 3, 4, 6, and multiply the p's by 6, we get a Nasik Square, but not in consecutive numbers."

If any one is satisfied that the foregoing is a proof or even an enunciation of the absolute negation in the text, I must leave the matter so, and can only say that I am unable to read the passage in that way.

* See additional note at the end of this paper.

When $n = 3$, the hyper-magic form is impossible except for equal elements.

The simply magic form is

$$\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\[4pt] \dfrac{4a_{13}-2a_{11}+a_{12}}{3} & \dfrac{a_{11}+a_{12}+a_{13}}{3} & \dfrac{4a_{11}-2a_{13}+a_{12}}{3} \\[8pt] \dfrac{2a_{11}+2a_{12}-a_{13}}{3} & \dfrac{2a_{11}+2a_{13}-a_{12}}{3} & \dfrac{2a_{12}+2a_{13}-a_{11}}{3}. \end{array}$$

Here the sum-values of the two principal diagonals, the rows and columns are the same. For the elements 1, 2, 3 ... 9, the form is one of the aspects of

$$\begin{array}{ccc} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{array}$$

We can make a complete set of parallel diagonals fulfil the weight condition by making

$$a_{11}+a_{12} = 2a_{13} \quad \text{or} \quad a_{12}+a_{13} = 2a_{11}.$$

When $n = 5$, the direct hyper-magic conditions are 20 in number, but not all independent. It is convenient to write $(p, q, r \ldots)$ for $a_{1p}+a_{2q}+a_{3r}+\ldots$, or simply $(p)_m$ when $p = q = r = \ldots$, and m is the number of elements involved. The order of the left-hand suffixes is supposed to remain unchanged.

The conditions are expressed directly by

$$\sum_{\mu=1}^{\mu=5} a_{1\mu} = \sum_{\mu=1}^{\mu=5} a_{2\mu} = \sum_{\mu=1}^{\mu=5} a_{3\mu} = \sum_{\mu=1}^{\mu=5} a_{4\mu} = \sum_{\mu=1}^{\mu=5} a_{5\mu} = W,$$

$$\begin{array}{ll} a_{31}+a_{41}+a_{51} = W-(1)_2, & a_{31}+a_{42}+a_{53} = W-(45), \\ a_{32}+a_{42}+a_{52} = W-(2)_2, & a_{32}+a_{43}+a_{54} = W-(51), \\ a_{33}+a_{43}+a_{53} = W-(3)_2, & a_{33}+a_{44}+a_{55} = W-(12), \\ a_{34}+a_{44}+a_{54} = W-(4)_2, & a_{34}+a_{45}+a_{51} = W-(23), \\ a_{35}+a_{45}+a_{55} = W-(5)_2, & a_{35}+a_{41}+a_{52} = W-(34), \end{array}$$

$$\begin{array}{l} a_{31}+a_{45}+a_{54} = W-(32), \\ a_{32}+a_{41}+a_{55} = W-(43), \\ a_{33}+a_{42}+a_{51} = W-(54), \\ a_{34}+a_{43}+a_{52} = W-(15), \\ a_{35}+a_{44}+a_{53} = W-(21). \end{array}$$

Writing C_{pq} for $a_{3p}-a_{3q}$, we get

$$(54)+(34)-(1)_2-(2)_2 = C_{13}-C_{52} = K_1,$$
$$(15)+(45)-(2)_2-(3)_2 = C_{24}-C_{13} = K_2,$$
$$(21)+(51)-(3)_2-(4)_2 = C_{35}-C_{24} = K_3,$$
$$(32)+(12)-(4)_2-(5)_2 = C_{41}-C_{35} = K_4,$$
$$(43)+(23)-(5)_2-(1)_2 = C_{52}-C_{41} = K_5.$$

But $-5a_{31}+W = -4a_{31}+a_{32}+a_{33}+a_{34}+a_{35} = 4C_{31}+3C_{53}+2C_{25}+C_{42}$
$$= 2K_2+K_3+2K_4$$

(because $C_{31}+C_{53}+C_{25}+C_{42}+C_{14} = 0$), and finally

$$a_{31} = a_{23}+a_{24}-a_{11}.$$

A hyper-magic square can always be varied, without losing its characteristic properties, by putting the last row first, or the last column first, or *vice versâ.** This is self-evident when the square is represented on an anchor ring, or is rolled round a cylinder. We can therefore obtain a_{32}, a_{33}, a_{34}, a_{35} by increasing the right-hand suffixes by unity successively, and rejecting multiples of 5 when the rule gives a suffix exceeding 5.

The elements of the last two rows are found by using the two next preceding rows in each case. Thus

$$a_{41} = (a_{25}+a_{21}-a_{13})+(a_{21}+a_{22}-a_{14})-a_{31} = W-a_{24}-a_{23}-a_{13}-a_{14}.$$

The general solution is sufficiently indicated by setting down the first and last columns, as follows:—

$$\begin{array}{ccccc}
a_{11} & . & . & . & a_{15} \\
a_{21} & . & . & . & a_{25} \\
a_{23}+a_{24}-a_{11} & . & . & . & a_{22}+a_{23}-a_{15} \\
W-a_{23}-a_{24}-a_{13}-a_{14} & . & . & . & W-a_{12}-a_{13}-a_{22}-a_{23} \\
a_{13}+a_{14}-a_{21} & . & . & . & a_{12}+a_{13}-a_{25},
\end{array}$$

with $$\sum_{\mu=1}^{\mu=5} a_{1\mu} = \sum_{\mu=1}^{\mu=5} a_{2\mu} = W.$$

* The definition of "carrés diaboliques" given by M. Lucas (*Récréations Math.*, Introduction, t. I., p. XVII.) is founded on this property.

4. If $n = 7$, these are the conditions :—

$$\sum_{\mu=1}^{\mu=7} a_{1\mu} = \sum_{\mu=1}^{\mu=7} a_{2\mu} = \sum_{\mu=1}^{\mu=7} a_{3\mu} = \sum_{\mu=1}^{\mu=7} a_{4\mu} = \sum_{\mu=1}^{\mu=7} a_{5\mu} = \sum_{\mu=1}^{\mu=7} a_{6\mu} = \sum_{\mu=1}^{\mu=7} a_{7\mu} = W,$$

and three other sets of conditions, the leading equations of which, expressed in the present notation, are

$$a_{51}+a_{61}+a_{71} = W-(1)_4, \quad a_{51}+a_{62}+a_{73} = W-(4567),$$
$$a_{51}+a_{67}+a_{76} = W-(5432).$$

The remaining conditions are formed by adding successively unity to each right-hand suffix, and rejecting multiples of 7 when a suffix exceeds 7.

Writing E_{pq} for $a_{5p}-a_{5q}$, we get, from the conditions,

$$(7654)+(3456)-(1)_4-(2)_4 = E_{13}-E_{72} = K_1,$$
$$(1765)+(4567)-(2)_4-(3)_4 = E_{24}-E_{13} = K_2,$$
$$(2176)+(5671)-(3)_4-(4)_4 = E_{35}-E_{24} = K_3,$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$(6543)+(2345)-(7)_4-(1)_4 = E_{72}-E_{61} = K_7.$$

Also
$$-7a_{51}+W = -6a_{51}+a_{52}+a_{53}+a_{54}+a_{55}+a_{56}+a_{57}$$
$$= 6E_{31}+5E_{53}+4E_{75}+3E_{27}+2E_{42}+E_{64}$$
$$= 3K_2+2K_3+4K_4+2K_5+3K_6,$$

by means of the identity

$$E_{13}+E_{35}+E_{57}+\ldots+E_{61} = 0.$$

Substituting the values of $K_2 \ldots K_6$ in terms of elements, we get

$$-7a_{51}+W = 7a_{11}-W+7(a_{21}+a_{22}+a_{27})-3W+7(a_{31}-a_{34}-a_{35})+W$$
$$+7(a_{41}+a_{42}+a_{47})-3W,$$

or
$$a_{51} = W-a_{11}-a_{21}-a_{22}-a_{27}-a_{31}+a_{34}+a_{35}-a_{41}-a_{42}-a_{47}.$$

The values of $a_{52} \ldots a_{57}$ follow by symmetry.

Making use now of the second, third, fourth, and fifth rows of elements, we get

$$a_{61} = a_{11}+a_{13}+a_{17}+a_{21}+a_{22}+a_{27}-a_{24}-a_{25}+a_{31}+a_{32}+a_{37}-a_{34}-a_{35}$$
$$+a_{41}+a_{42}+a_{47}-W.$$

The full solution is sufficiently shown by setting down the first and last columns, thus :—

$$\begin{array}{ccccccc}
a_{11} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{17} \\
a_{21} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{27} \\
a_{31} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{37} \\
a_{41} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{47}
\end{array}$$

$$W-a_{11}-a_{21}-a_{22}-a_{27}-a_{31}+a_{34}+a_{36}-a_{41}-a_{42}-a_{47} \ldots W-a_{17}-a_{27}-a_{21}-a_{26}-a_{37}+a_{33}+a_{34}-a_{47}-a_{41}-$$

$$\left.\begin{array}{l} a_{11}+a_{13}+a_{17}+a_{21}+a_{22}+a_{27}-a_{24}-a_{25} \\ +a_{31}+a_{32}+a_{37}-a_{34}-a_{35}+a_{41}+a_{42}+a_{47}-W \end{array}\right\} \ldots \left\{\begin{array}{l} a_{17}+a_{11}+a_{16}+a_{27}+a_{21}+a_{26}-a_{23}-a_{24} \\ +a_{37}+a_{31}+a_{36}-a_{33}-a_{34}+a_{47}+a_{41}+a_{46}-W \end{array}\right.$$

$$W-a_{41}-a_{31}-a_{32}-a_{37}-a_{21}+a_{24}+a_{26}-a_{11}-a_{12}-a_{17} \ldots W-a_{47}-a_{37}-a_{31}-a_{36}-a_{27}+a_{23}+a_{24}-a_{17}-a_{11}-$$

with $$\sum_{\mu=1}^{\mu=7} a_{1_\mu} = \sum_{\mu=1}^{\mu=7} a_{2_\mu} = \sum_{\mu=1}^{\mu=7} a_{3_\mu} = \sum_{\mu=1}^{\mu=7} a_{4_\mu} = W.$$

5. We can now see the form of the general solution for $n = 2m+1$.

Using the same notation, and writing Pab for $a_{2m-1,a}-a_{2m-1,b}$, we get, from the hyper-magic conditions,

$$(2m+1,\ 2m,\ 2m-1 \ldots 4)+(345 \ldots 2m)-(1)_{2m-2}-(2)_{2m-2}$$
$$= P_{13}-P_{2m+1,2} = K_1,$$

$$(1,\ 2m+1,\ 2m\ \ldots\ 5)+(456 \ldots 2m+1)-(2)_{2m-2}-(3)_{2m-2}$$
$$= P_{24}-P_{13} = K_2,$$

... ...

$$(2m,\ 2m-1,\ 2m-2 \ldots 3)+(234 \ldots 2n-1)-(2m+1)_{2m-2}-(1)_{2m-2}$$
$$= P_{2m+1,2}-P_{2m,1} = K_{2m+1},$$

and also

$$-(2m+1)\,a_{2m-1,1}+W = 2mP_{31}+(2m-1)\,P_{53}+(2m-2)\,P_{75}+\ldots$$
$$\ldots+(m+1)\,P_{2m+1,2m-1}+mP_{2,2m+1}+(m-1)\,P_{42}$$
$$+(m-2)\,P_{64}+\ldots+P_{2m,2m-2}$$
$$= mK_2+(m-1)\,K_3+2\,(m-1)\,K_4+2\,(m-2)\,K_5$$
$$+3\,(m-2)\,K_6+3\,(m-3)\,K_7+\ldots$$
$$\ldots+2\,(m-1)\,K_{2m-2}+(m-1)\,K_{2m-1}+mK_{2m}.$$

The last right-hand equivalent is verified by adding the zero value $m(P_{31}+P_{42}+\ldots+P_{2,2m+1})$, and replacing $K_2 \ldots K_{2m}$ by their P-values.

The values of $a_{2m-1,2} \ldots a_{2m-1,2m+1}$ are obtained by symmetry, and the remaining rows also by repetition of same process applied to the $2m-2$ next preceding rows in each case.

6. In order, however, to reach definite conclusions, we must go into a little further detail.

It will be shown that the numerical coefficients of the literal elements of any one of the first $2m-2$ rows in the value of $a_{2m-1,1}$ have the same residues relative to $2m+1$ as modulus. To this end, the result of the last article is written more fully, as follows :—

$$
\left.
\begin{aligned}
&-(2m+1)\,a_{2m-1,1}+W\\
&= \quad m\,\{(1,2m+1,2m\ldots5)+(4,5,6\ldots2m+1)-(2)_{2m-2}-(3)_{2m-2}\}\\
&\quad+(m-1)\,\{(2,1,2m+1\ldots6)+(5,6,7\ldots2m+1,1)-(3)_{2m-2}-(4)_{2m-2}\}\\
&\quad+2(m-1)\,\{3,2,1\ldots7)+(6,7,8\ldots1,2)-(4)_{2m-2}-(5)_{2m-2}\}\\
&\quad+\;\ldots\;\ldots\;\ldots\;\ldots\;\ldots\;\ldots\;\ldots\;\ldots\;\ldots\\
&\quad+(m-1)\,\{(2m-2,2m-3,2m-4\ldots1)+(2m+1,1,2\ldots2m-3)\\
&\qquad\qquad\qquad\qquad-(2m-1)_{2m-2}-(2m)_{2m-2}\}\\
&\quad+m\,\{(2m-1,2m-2,2m-3\ldots2)+(1,2,3\ldots2m-2)\\
&\qquad\qquad\qquad\qquad-(2m)_{2m-2}-(2m+1)_{2m-2}\}
\end{aligned}
\right\}
\qquad\ldots\ldots\ldots\ldots(B).
$$

For uniformity of law, we may consider unity in the first row as equivalent to $2m+2$.

Let p be a suffix of the series $(2m+2, 2m+1 \ldots 5)$, and let r be the $r+1^{\text{th}}$ suffix in the column of which the leading term is p. Then

$$p+r = r+\rho\,(2m+1) \quad\ldots\ldots\ldots\ldots\ldots\ldots(\alpha),$$

where ρ is zero or unity. Similarly, let q be a suffix of the series $(4, 5, 6 \ldots 2m+1)$ of the same rank (say σ) as p, and let r be the $s+1^{\text{th}}$ suffix of the column whose leading term is q. Then

$$q+s = r+\rho'\,(2m+1) \quad\ldots\ldots\ldots\ldots\ldots\ldots(\beta),$$

when ρ' is zero or unity. Also we have

$$p+q = 2m+6 \quad\ldots\ldots\ldots\ldots\ldots\ldots(\gamma).$$

If r is odd, the numerical coefficient, outside the brackets of the $r+1^{\text{th}}$ row, is

$$\frac{r+1}{2}\left(m-\frac{r+1}{2}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots(\delta);$$

if r is even, the coefficient is

$$\frac{r+2}{2}\left(m-\frac{r}{2}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots(\epsilon).$$

The same rules apply to s. For $r = 2m-1$ or $2m$, the value of (δ) or (ϵ) is zero.

There are several cases to consider.

I., r and s both odd: the coefficient of a_{sr} in the right-hand member of (B), is, by (α), (β), (γ), and (δ),

$$\frac{r-p+1+\rho\,(2m+1)}{2}\cdot\frac{(1-\rho)(2m+1)+p-r-2}{2}$$
$$+\frac{r+p-4-(1-\rho')(2m+1)}{2}\cdot\frac{(2-\rho')(2m+1)-p-r+3}{2}$$
$$-\frac{(r-1)(2m-r+2)}{2},$$

the last term being due to terms $(r)_{2m-2}$.

Since ρ, ρ' are in this case both zero or both unity, we write ρ for ρ', and the expanded coefficient is

$$\tfrac{1}{4}\left\{-2\,(1-\rho)^2(2m+1)^2+\rho\,(4-4r)(2m+1)+(2r+2p-8)(2m+1)\right.$$
$$\left.-2\,(p-2)(p-3)\right\}.$$

The residue, relative to $2m+1$ as modulus, is independent of r, since the explicit multiplier of $2m+1$ is integer.

II., r and s are both even: the coefficient of a_{sr} is by (α), (β), (γ) and (ϵ),

$$\frac{r-p+2+\rho\,(2m+1)}{2}\cdot\frac{(1-\rho)(2m+1)+p-r-1}{2}$$
$$+\frac{r+p-3-(1-\rho')(2m+1)}{2}\cdot\frac{(2-\rho')(2m+1)-r-p+4}{2}$$
$$-\frac{(r-1)(2m-r+2)}{2}.$$

We may write ρ for ρ', since they have the same values, and the expanded expression is

$$\tfrac{1}{4}\left\{-2\,(1-\rho)^2(2m+1)^2+\rho\,(4-4r)(2m+1)+(2r+2p-6)(2m+1)\right.$$
$$\left.-2\,(p-2)(p-3)\right\}.$$

The residue, relative to $2m+1$ as modulus, is independent of r, and the same as before.

In the cases III., r odd and s even, IV., r even, s odd, we may put $1-\rho$ for ρ'. The coefficient of $a_{\sigma\tau}$ is, in both cases,

$$\tfrac{1}{4}\left\{-2\rho^2(2m+1)^2+\rho(4p-10)(2m+1)-2(p-2)(p-3)\right\}.$$

The residue is the same, relative to $2m+1$ as modulus.

7. It appears then that

$$-(2m+1)\,a_{2m-1,1}=(2m+1)\,L+\mu W,$$

where L is a linear and integral function of the elements of the first $2m-2$ rows, and

$$\mu=-1-\sum_{p=5}^{p=2m+2}\frac{p-2\,.\,p-3}{2}=-\frac{2m-1\,.\,2m\,.\,2m+1}{2\,.\,3}.$$

Hence, if all the elements are to be integers, and $2m+1$ is a multiple of 3, W must also be a multiple of 3. This condition is satisfied when the elements are in arithmetical progression, and in fact hyper-magic squares can be formed in this case. Yet there is a well-known fundamental distinction between the cases of orders prime and not prime to 3, as to manner of construction.

This difference is due to the circumstance that if we have $2m+1$ distinct literal elements repeated $2m+1$ times, we cannot form a hyper-magic square with them when the order is a multiple of 3. The conditions

$$\sum_{\mu=1}^{\mu=2m+1}a_{1\mu}=\sum_{\mu=1}^{\mu=2m+1}a_{2\mu}=\ldots=\sum_{\mu=1}^{\mu=2m+1}a_{2m-2,\mu}=W$$

are to be satisfied.

8. I pass now to the case of an even order. For the order 4 the general form is

$$\begin{array}{cccc}
a_1, & a_2, & a_3, & a_4, \\[4pt]
\dfrac{W}{2}-a_1+s, & \dfrac{W}{2}-a_2-s, & \dfrac{W}{2}-a_3+s, & \dfrac{W}{2}-a_4-s, \\[8pt]
\dfrac{W}{2}-a_3, & \dfrac{W}{2}-a_4, & \dfrac{W}{2}-a_1, & \dfrac{W}{2}-a_2, \\[8pt]
a_3-s, & a_4+s, & a_1-s, & a_2+s,
\end{array}$$

of course, with the condition

$$a_1+a_2+a_3+a_4 = W.$$

For integral elements, therefore, W must be even. This condition is fulfilled by numbers in arithmetical progression. A number s is introduced, for which there is no equivalent when the order is odd. The next even order, 6, shows the same peculiarity. We have

$$\begin{array}{cccccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ \ldots & \ldots & \ldots & \ldots & & \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}. \end{array}$$

The sums of these rows must satisfy the weight condition, and then

$$\begin{array}{ll} a_{41}+a_{51}+a_{61} = W-(1)_3, & a_{41}+a_{52}+a_{63} = W-(456), \\ a_{42}+a_{52}+a_{62} = W-(2)_3, & a_{42}+a_{53}+a_{64} = W-(561), \\ \ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots\ \ldots \\ a_{46}+a_{56}+a_{66} = W-(6)_3, & a_{46}+a_{51}+a_{62} = W-(345), \end{array}$$

$$a_{41}+a_{56}+a_{65} = W-(432),$$
$$a_{42}+a_{51}+a_{66} = W-(543),$$
$$\ldots\ \ldots\ \ldots\ \ldots$$
$$a_{46}+a_{55}+a_{64} = W-(321),$$

whence we get

$$\begin{array}{ll} a_{41}-a_{43}+a_{31}-a_{53} = (654)-(1)_3, & a_{41}-a_{45}+a_{51}-a_{53} = (234)-(1)_3, \\ a_{42}-a_{44}+a_{52}-a_{53} = (165)-(2)_3, & a_{42}-a_{46}+a_{53}-a_{51} = (345)-(2)_3, \\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots & \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\ a_{46}-a_{42}+a_{56}-a_{31} = (543)-(6)_3, & a_{46}-a_{44}+a_{56}-a_{55} = (123)-(6)_3; \end{array}$$

and then, writing P_{ab} for $a_{4a}-a_{4b}$,

$$a_{41}-a_{45}+a_{42}-a_{46} = (654)+(345)-(1)_3-(2)_3 = P_{15}+P_{26} = K_1,$$
$$a_{42}-a_{44}+a_{43}-a_{41} = (165)+(456)-(2)_3-(3)_3 = P_{24}+P_{31} = K_2,$$
$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$
$$a_{46}-a_{42}+a_{41}-a_{45} = (543)+(234)-(6)_3-(1)_3 = P_{62}+P_{15} = K_6,$$

and $$\sum_{\mu=1}^{\mu=6} a_{1\mu} = \sum_{\mu=1}^{\mu=6} a_{2\mu} = \ldots = \sum_{\mu=1}^{\mu=6} a_{6\mu} = W.$$

Also we have

$$P_{31} = P_{31},$$
$$P_{51} = P_{31}+P_{53},$$
$$P_{21} = P_{31}+P_{53}+P_{26}+P_{65},$$
$$P_{41} = P_{31}+P_{53}+P_{26}+P_{42}+P_{65},$$
$$P_{61} = P_{31}+P_{53}+P_{26}+P_{42}+P_{64}+P_{65},$$

whence $\quad -6a_{41}+W = 5P_{31}+4P_{53}+3P_{26}+2P_{42}+P_{64}+3P_{65};$

similarly $\quad -6a_{42}+W = 5P_{42}+4P_{64}+3P_{31}+2P_{53}+P_{15}+3P_{16},$

and $\quad -6\,(a_{41}+a_{42})+2W = 8P_{31}+6P_{53}+3P_{26}+7P_{42}+5P_{64}+4P_{15},$

which can be expressed in terms of the K's.

For, taking

$$pK_1+qK_2+rK_3+sK_4+tK_5+uK_6+k\,(P_{31}+P_{53}+P_{15})+l(P_{42}+P_{26}+P_{64}),$$

and identifying the expression with the value of $-6\,(a_{41}+a_{42})+2W$, we may make $p=u=0$ (because $K_1+K_3+K_5 = K_2+K_4+K_6 = 0$), so that

$$k+q=8,\quad k+s-r=6,\quad k-t=4,\quad l+r-q=7,\quad l=3,$$
$$l+t-s=5,$$

the solution of which is

$$q=0,\quad k=8,\quad t=4,\quad s=2,\quad r=4;$$

and therefore $\quad -3\,(a_{41}+a_{42})+W = 2K_3+K_4+2K_5;$

similarly $\quad -3\,(a_{42}+a_{43})+W = 2K_4+K_5+2K_6,$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

These equations show that one of the elements of the fourth row remains undetermined.

Substituting the values of the K's, we get

$$-(a_{41}+a_{42}) = a_{11}+a_{12}+a_{21}+a_{22}-a_{24}-a_{25}+a_{31}+a_{32}-a_{34}-a_{35}-\frac{2W}{3}.$$

Put $\quad -(\delta+a_{41}) = a_{11}+a_{21}-a_{24}+a_{31}-a_{34}-\frac{W}{3},$

and then we have

$$a_{41} = -a_{11} - a_{21} + a_{24} - a_{31} + a_{34} + \frac{W}{3} - \delta,$$

$$a_{42} = -a_{12} - a_{22} + a_{25} - a_{32} + a_{35} + \frac{W}{3} + \delta,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$a_{46} = -a_{16} - a_{26} + a_{23} - a_{36} + a_{33} + \frac{W}{3} + \delta.$$

This shows that for integral elements the weight must be a multiple of 3.

But W must also be even; for, if we take as given the elements of the first four rows, we get

$$\begin{aligned} -6a_{51} + W &= 5(a_{52} - a_{51}) + 4(a_{53} - a_{52}) + 3(a_{54} - a_{53}) \\ &\qquad + 2(a_{55} - a_{54}) + a_{56} - a_{55} \\ &= 5[(3456) - (2)_4] + 4[(4561) - (3)_4] \\ &\quad + 3[(5612) - (4)_4] + [(6123) - (5)_4] + (1234) - (6)_4 \\ &= W - 6a_{12} + 2W - 6a_{22} - 6a_{23} + 3W - 6a_{32} - 6a_{33} - 6a_{34} \\ &\qquad + 4W - 6a_{42} - 6a_{43} - 6a_{44} - 6a_{45}, \end{aligned}$$

or $$a_{51} = a_{12} + a_{22} + a_{23} + a_{32} + a_{32} + a_{33} + a_{34} - a_{41} - a_{46} - \frac{W}{2}.$$

Substituting for $a_{41} + a_{46}$,

$$a_{51} = a_{11} + a_{12} + a_{16} + a_{21} + a_{22} + a_{26} - a_{24} + a_{31} + a_{32} + a_{36} - \tfrac{7}{6}W.$$

The first column of the general form of solution is therefore

$$\begin{gathered} a_{11}, \\ a_{21}, \\ \cdots \quad \cdots \quad \cdot \quad \cdot \quad \cdot \\ a_{31}, \\ -a_{11} - a_{21} + a_{24} - a_{31} + a_{34} + \frac{W}{3} - \delta, \\ a_{11} + a_{12} + a_{16} + a_{21} + a_{22} + a_{26} - a_{24} + a_{31} + a_{32} + a_{36} - \tfrac{7}{6}W, \\ -a_{11} - a_{12} - a_{16} - a_{21} - a_{22} - a_{26} - a_{31} - a_{32} - a_{36} - a_{34} + \tfrac{11}{6}W + \delta. \end{gathered}$$

The other columns are obtained by successively adding unity to the right-hand suffixes and changing the signs of δ.

9. Let $n = 2m$, and denote $a_{2m-2,\mu} - a_{2m-2,\nu}$ by $P_{\mu\nu}$.

Then, analogously to previous cases,

$$P_{13}+P_{2,2m} = (2m, 2m-1 \dots 4) - (1)_{2m-3} + (3, 4 \dots 2m-1) - (2)_{2m-3} = K_1,$$

$$P_{24}+P_{31} = (1, 2m \dots 5) - (2)_{2m-3} + (4, 5 \dots 2m) - (3)_{2m-3} = K_2,$$

$$P_{35}+P_{42} = (2, 1 \dots 6) - (3)_{2m-3} + (5, 6 \dots 2m, 1) - (4)_{2m-3} = K_3,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$P_{2m,2}+P_{1,2m-1} = (2m-1, 2m-2 \dots 3) - (2m)_{2m-3} + (2, 3 \dots 2m-2) - (1)_{2m-3} = K_{2m};$$

also, since

$$P_{31} = P_{31},$$

$$P_{51} = P_{31}+P_{53},$$

$$\dots \quad \dots \quad \dots$$

$$P_{21} = P_{31}+P_{53}+ \dots + P_{2,2m}+P_{2m,2m-1},$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$P_{2m,1} = P_{31}+P_{53}+ \dots P_{2m,2m-2}+P_{2m,2m-1},$$

we have

$$-2ma_{2m-2,1}+W = (2m-1)\,P_{31}+(2m-2)\,P_{53}+ \dots +(m+1)\,P_{2m-1,2m-3}$$
$$+mP_{2,2m}+(m-1)\,P_{42}+ \dots +P_{2m,2m-2}+mP_{2m,2m-1},$$

and the values of $a_{2m-2,2} \dots a_{2m-2,2m}$ follow by symmetry; so that we have

$$-2m\,(a_{2m-2,1}+a_{2m-2,2})+2W$$
$$= (3m-1)\,P_{31}+(3m-3)\,P_{53}+(3m-5)\,P_{75}+ \dots$$
$$+(m+3)\,P_{2m-1,2m-3}+(m+1)\,P_{1,2m-1}+mP_{2,2m}$$
$$+(3m-2)\,P_{42}+(3m-4)\,P_{64}+ \dots +(m+2)\,P_{2m,2m-2}.$$

Identifying the right-hand member with

$$p_1K_1+p_2K_2+p_3K_3+ \dots +p_{2m}K_{2m}$$
$$+k\,(P_{31}+P_{53}+ \dots +P_{2m-1,2m-3}+P_{1,2m-1})+l\,(P_{2,2m}+P_{42}+ \dots +P_{2m,2m-2}),$$

we get

$$k+p_2-p_1 = 3m-1, \quad k+p_4-p_3 = 3m-3, \quad \dots \quad k+p_{2m}-p_{2m-1} = m+1,$$
$$l+p_3-p_2 = 3m-2, \quad l+p_5-p_4 = 3m-4, \quad \dots \quad l+p_{2m-1}-p_{2m-2} = m+2,$$
$$l+p_1-p_{2m} = m.$$

The solution is (when, by means of the two zero periods of the K's, we make $p_1 = p_{2m} = 0$), $k = 3m-1$, $l = m$,

$$p_2 = 0,\quad p_3 = 2m-2,\quad p_5 = 4m-8,\quad p_7 = 6m-18,\ \dots\ p_{2m-1} = 2m-2.$$

$$p_4 = 2m-4,\quad p_6 = 4m-12,\quad p_8 = 6m-24,\ \dots\ p_{2m-2} = 2m-4;$$

whence

$$\begin{aligned}
&-m\,(a_{2m-2,1}+a_{2m-2,2})+W\\
&= (m-1)\left\{(2,1,2m\dots 6)-(3)_{2m-3}+(5,6\dots 2m,1)-(4)_{2m-3}\right\}\\
&\quad+(m-2)\left\{(3,2,1\dots 7)-(4)_{2m-3}+(6,7\dots 1,2)-(5)_{2m-3}\right\}\\
&\quad+2(m-2)\left\{(4,3,2\dots 8)-(5)_{2m-3}+(7,8\dots 2,3)-(6)_{2m-3}\right\}\\
&\quad+\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\\
&\quad+(m-1)\left\{(2m-2,2m-3\dots 1,2)-(2m-1)_{2m-3}\right.\\
&\qquad\qquad\left.+(1,2,3\dots 2m-3)-(2m)_{2m-3}\right\}.
\end{aligned}$$

For uniformity of law we may write $2m+2$, $2m+1$ for 2, 1 in the first row of the right-hand member.

Let p be the σ^{th} suffix in the series $(2m+2\dots 6)$, and let q be the σ^{th} suffix in the series $(5, 6\dots 2m-1)$. Then, for the suffix r, in the $r+1^{\text{th}}$ place of the column to which p belongs, and in the $s+1^{\text{th}}$ place in the column to which q belongs, we have

$$p+r = r+\rho\,(2m),\quad q+s = r+\rho'\,(2m),\quad p+q = 2m+7,$$

when ρ, ρ' are zero or unity. Observe also that the coefficient outside the brackets of the $r+1^{\text{th}}$ row is $\frac{r+1}{2}\left(m-\frac{r+3}{2}\right)$ if r is odd, and $\frac{r+2}{2}\left(m-\frac{r+2}{2}\right)$ if r is even. The coefficient of $(r)_{2m-3}$ is

$$-\frac{(r-2)(2m-r+1)}{2}.$$

Now r and s are not of the same parity, because p and q are not. If r is even and s odd, the coefficient of $a_{s,r}$ on right-hand side of the equation is

$$\frac{r-p+2+\rho\,(2m)}{2}\cdot\frac{(1-\rho)(2m)-r+p-2}{2}$$

$$+\frac{r+p-6+(\rho'-1)(2m)}{2}\cdot\frac{(2-\rho')(2m)-r-p+4}{2}$$

$$-\frac{(r-2)(2m-r+1)}{2}.$$

Expanding this, we find, after taking away the multiple of m, as it appears explicitly, a residue $-\frac{p-3\,.\,p-4}{2}$, which is independent of r; the result is the same if r is odd and s even. Hence

$$-m\,(a_{2m-2,1}+a_{2m-2,2}) = mL+\mu W,$$

where L is a linear and integral function of the elements, and

$$\mu = -1-\sum_{p=6}^{p=2m+2}\frac{p-3\,.\,p-4}{2} = -\frac{2m-2\,.\,2m-1\,.\,2m}{2\,.\,3}.$$

Hence, if the elements are all integral and m is a multiple of 3, W must also be a multiple of 3. The conditions

$$\sum_{\mu=1}^{\mu=2m} a_{1,\mu} = \dots = \sum_{\mu=1}^{\mu=2m} a_{2m-3,\mu} = W$$

must be satisfied.

10. To show that W must also be even, recourse must be had to the value of $a_{2m-1,1}$, when the elements of the first $2m-2$ rows are assumed to be given. We may take the form for n generally, viz.,

$$\left.\begin{array}{ccc} a_{1,1}, & a_{1,2} & \dots\ a_{1,n} \\ \dots & \dots & \dots \\ a_{n,1}, & a_{n,2} & \dots\ a_{n,n} \end{array}\right\} \qquad \dots\dots\dots\dots\dots\dots\dots\dots (\mathrm{C}).$$

Then

$$\begin{aligned} a_{n-1,1}-a_{n-1,n} &= (2, 3, 4 \dots n-1)-(1)_{n-2} = K_1, \\ a_{n-1,2}-a_{n-1,1} &= (3, 4, 5 \dots n) \quad -(2)_{n-2} = K_2, \\ \dots \quad & \dots \quad \dots \quad \dots \\ a_{n-1,n}-a_{n-1,n-1} &= (1, 2, 3 \dots n-2)-(n)_{n-2} = K_n. \end{aligned}$$

Because $a_{n-1,2} = K_2+a_{n-1,1}$, $a_{n-1,3} = K_3+K_2-a_{n-1,1}$,

$$a_{n-1,4} = K_4+K_3+K_2-a_{n-1,1}, \text{ \&c.},$$

and

$$a_{n-1,1}+a_{n-1,2}+\dots+a_{n-1,n} = w,$$

we get $-na_{n-1,1}+W = (n-1)\,K_2+(n-2)\,K_3+(n-3)\,K_4+\dots+K_n$.

This may be written more conveniently

$$\begin{aligned} -na_{n-1,1}+W = \; & (n-1)\{(3, 4, 5 \dots n)-(2)_{n-2}\} \\ & +(n-2)\{(4, 5, 6 \dots 1)-(3)_{n-2}\} \\ & +(n-3)\{(5, 6, 7 \dots 2)-(4)_{n-2}\} \\ & + \quad \dots \quad \dots \quad \dots \quad \dots \\ & + \quad \{(1, 2, 3 \dots n-2)-(n)_{n-2}\}. \end{aligned}$$

Let p be the σ^{th} suffix of the set $(3, 4, 5 \ldots n)$, and let r be the suffix in the $r+1^{\text{th}}$ row on the right-hand side of the equation, and in the column under p. Then

$$p+r = \tau+\rho n,$$

where ρ is zero or unity. The corresponding multiplier outside the brackets is $n-\tau-1$, and the whole coefficient of $a_{\sigma\tau}$ on the right-hand side of the equation is $(n-\tau-\rho n+p-1)-(n-\tau+1)$, and this is congruent with $p-2$ as to the mod n.

Hence $$-na_{n-1,1} = nL+\mu W,$$

where L is an integral linear function of the elements of the first $n-2$ rows of (C), and

$$\mu = \sum_{p=3}^{p=n}(p-2)-1 = \frac{n-2\,.\,n-1}{2}-1 = \frac{n(n-3)}{2}\,.$$

It follows that for integer elements W must be even, if n is even.

Hence a hyper-magic square of even order cannot be formed with integer elements unless the weight is even, nor if the order is a multiple of 3, unless the weight is also a multiple of 3.

The elements $1, 2, 3 \ldots 4(2m+1)^2$ give the weight

$$= (2m+1)\left[4(2m+1)^2+1\right],$$

which is odd. Consequently, a hyper-magic square with these elements is impossible in every case.

Moreover, it is not possible to form an oddly even hyper-magic square with integer elements in arithmetical progression, as $a+d$, $a+2d$, &c. For such a square is the sum of two squares, one of them having equal elements, and the other having the elements $1, 2, 3 \ldots 4(2m+1)^2$, each multiplied by the common difference, and this is the case though we make the weight even by appropriate values of a and d.

Additional Note.

[I did not think it necessary to justify my use of the names "magic square" and "hyper-magic square" for squares filled up with any numbers fulfilling the usual magical conditions as to summation. But some mathematicians still insist on a narrow meaning, and,

therefore, I now add several extracts which bear me out in a more liberal application of the terms.

It is quite true that the earlier (and, as I think, unfortunately, some later) definitions require a magic square to have for elements the natural numbers from 1 to n^2, where n is the root or order. But it was found unnecessary and mathematically undesirable so to restrict the meaning. Thus, Schottus (1664, *Curiosa Technica*, Lib. XI., Cap. XIV., quoted by Günther) formulates the problem as follows: "Numeros quoscunque quadratos ita in quadrata disponere, ut quævis series additæ, sive transversim sumantur sive a summo deorsum sive decussatim seu diagonaliter semper eandem summam conficiant." The historical notice connected with M. Sauveur's paper (1710, *Mém. de l'Académie Royale des Sciences*) contains the following passage: "De tout cela il suit qu'au lieu qu'on prenoit pour la construction des Quarrés Magiques que des nombres en progression arithmétique et même naturelle, la choix est beaucoup plus libre qu'on ne pensoit. C'est telle liberté, reconnue par M. Sauveur dans toute son étendue et avec les seules restrictions absolument necessaires, qui lui a fait naître la pensée de construire les Quarrés Magiques par lettres, c'est-à-dire d'une manière beaucoup plus générale que l'on n'a jamais fait et aussi générale qu'il soit possible. Car dès que les nombres ont quelque chose en général et d'indéterminé, les lettres sont propres à exprimer toute leur généralité et leur indétermination."

Then, in Hutton's edition of *Ozanam's Recreations* (1803) we find that "the name 'magic square' is given to a square divided into several other small equal squares or cells, filled up with the terms of any progression of numbers, but generally an arithmetical one, in such a manner that those in each band, whether horizontal or vertical or diagonal, shall always form the same sum." In the *Penny Cyclopædia*, it is, I suppose, Professor de Morgan who writes: "Magic square.—This term is applied to a set of numbers arranged in a square, in such a manner that the vertical, horizontal, and diagonal columns shall give the same sums." There are intermediate definitions. In fact, two things strike anyone who looks into the history of the subject: (1) the vacillation and ambiguity of definition, (2) the frequent reproduction and development of old methods of formation without due recognition of previous results.]

Thursday, December 8th, 1892.

A. B. KEMPE, Esq., F.R.S., President, in the Chair.

Messrs. H. G. Dawson, M.A., Fellow of Christ's College, Cambridge, W. J. Greenstreet, M.A., formerly of St. John's College, Cambridge, and W. Welsh, M.A., Fellow and Mathematical Tutor of Jesus College, Cambridge, were elected members.

The Auditor, Mr. Heppel, having read his report, upon the motion of Professor Greenhill, seconded by Lieut.-Col. Cunningham, the Treasurer's report was adopted, and Mr. Heppel thanked for the trouble he had taken.

The following communications were made:—

On a Theorem in Differentiation, and its Application to Spherical Harmonics: Dr. Hobson.

On Cauchy's Condensation Test for the Convergency of Series: Dr. M. J. M. Hill.

Additional Note on Secondary Tucker Circles: Mr. J. Griffiths.

Notes on Determinants: Mr. J. E. Campbell.

A Geometrical Note: Mr. R. Tucker.

The President (Major MacMahon in the chair) made an impromptu communication upon a problem which he thought to be subsidiary to that of the "Stamp-folding" Problem.

The following presents were received:—

"Vector Algebra and Trigonometry," by R. Baldwin Hayward; 8vo, 1892. From the Author.

"Zeittafeln zur Geschichte der Mathematik, Physik, und Astronomie, bis zum Jahre 1500," von Dr. Felix Müller; 8vo, Leipzig, 1892.

"Beiblätter zu den Annalen der Physik und Chemie," Band XVI., Stück 10.

"Proceedings of the Royal Society," Vol. LII., No. 316.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig," 1892, III.

"Bulletin of the New York Mathematical Society," Vol. II., No. 2.

"Bulletin de la Société Mathématique de France," Tome XX., No. 5.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXVI., 3me Livraison; Harlem.

"Entwurf einer neuen Integralrechnung auf Grund der Potenzial-, Logarithmal-, und Numeralrechnung," von Dr. Julius Bergbohm; Pamphlet, 8vo, Leipzig, 1892.

"Kansas University Quarterly," Vol. I., No. 2; October, 1892.

"Bulletin des Sciences Mathématiques," Tome XVI.; October, 1892.

"Rendiconti del Circolo Matematico di Palermo," Tomo VI., Fasc. 5.

"Bestimmung der Trägheitsmomente des menschlichen Körpers und seiner Glieder," von W. Braune und O. Fischer, No. VIII. des XVIII. Bandes der Abhandlungen der mathematisch-physischen Classe der K.-S. Gesells. der Wissenchaften zu Leipzig.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. I., Fasc. 8–9, 2° Semestre; Roma, 1892.

"Educational Times," December, 1892.

"Annales de la Faculté des Sciences de Toulouse," Tome VI., Fasc. 3; 1892.

"Indian Engineering," Vol. XII., Nos. 18, 19, 20.

"Invention," Vol. XIV., No. 706, N.S.

On a Theorem in Differentiation, and its application to Spherical Harmonics. By E. W. HOBSON. Received and read December 8th, 1892.

It has been shown by Clebsch,* in a paper entitled "Ueber eine Eigenschaft der Kugelfunctionen," that, if $f_n(x, y, z)$ denote any rational homogeneous function of x, y, z of degree n, the expression

$$f_n - \frac{r^2 \nabla^2 f_n}{2 \, . \, 2n-1} + \frac{r^4 \nabla^4 f_n}{2 \, . \, 4 \, . \, 2n-1 \, . \, 2n-3} - \dots$$

is a spherical harmonic, where $r^2 = x^2+y^2+z^2$, and ∇^2 is Laplace's operator. The consideration of this theorem has led me to a theorem in differentiation which it is the object of the present communication to investigate and to apply to the theory of spherical harmonics.

1. Let $f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ denote a rational homogeneous function of degree n of the three operators, and suppose it required to find an expression for $f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r}$, r denoting $(x^2+y^2+z^2)^{\frac{1}{2}}$. It is clear that the required expression is of the form

$$(-1)^n . 1 . 3 . 5 \dots (2n-1) \left[\frac{f_n(x, y, z)}{r^{2n+1}} + \frac{f_{n-2}}{r^{2n-1}} + \frac{f_{n-4}}{r^{2n-3}} + \dots\right],$$

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* *Crelle's Journal*, Vol. LX., 1862.

when $f_{n-2}, f_{n-4} \ldots$ are certain functions of x, y, z, to be determined, of degrees given by the suffixes.

This expression must be a spherical harmonic of degree $-(n+1)$, and the corresponding positive harmonic of degree n is obtained by multiplying the expression by r^{2n+1}, thus

$$f_n + r^2 f_{n-2} + r^4 f_{n-4} + \ldots$$

is a spherical harmonic.

Now
$$\begin{aligned} \nabla^2 (r^2 f_{n-2}) &= 2\,(2n-1)\; f_{n-2} + r^2 \nabla^2 f_{n-2}, \\ \nabla^2 (r^4 f_{n-4}) &= 4\,(2n-3)\; r^2 f_{n-4} + r^4 \nabla^2 f_{n-4}, \\ \nabla^2 (r^6 f_{n-6}) &= 6\,(2n-5)\; r^4 f_{n-6} + r^6 \nabla^2 f_{n-6}, \\ \ldots \quad & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots; \end{aligned}$$

hence the condition

$$\nabla^2 (f_n + r^2 f_{n-2} + r^4 f_{n-4} + \ldots) = 0$$

is satisfied if

$$f_{n-2} = -\frac{1}{2.2n-1} \nabla^2 f_n, \qquad f_{n-4} = -\frac{1}{4.2n-3} \nabla^2 f_{n-2},$$

$$f_{n-6} = -\frac{1}{6.2n-5} \nabla^2 f_{n-4}, \text{ \&c.};$$

we thus have $\left(1 - \dfrac{r^2 \nabla^2}{2.2n-1} + \dfrac{r^4 \nabla^4}{2.4.2n-1.2n-3} - \ldots\right) f_n.$

The values of $f_{n-2}, f_{n-4} \ldots$ thus found are the only possible values which can make $f_n + r^2 f_{n-2} + r^4 f_{n-4} + \ldots$ a spherical harmonic. For if any other values could be found for these functions, we should have two spherical harmonics of degree n whose difference would be a multiple of r^2, say $r^2 u_{n-2}$; this would itself be a spherical harmonic. Now

$$\nabla^2 (r^2 u_{n-2}) = 2\,(2n-1)\; u_{n-2} + r^2 \nabla^2 u_{n-2};$$

thus it is impossible that $r^2 u_{n-2}$ should be a spherical harmonic, unless u_{n-2} is a multiple of r^2, say $r^2 v_{n-4}$. Similarly it may be shown that v_{n-4} is a multiple of r^2, and ultimately that u_{n-2} vanishes. Thus the system of values found for $f_{n-2}, f_{n-4} \ldots$ is the only possible one. We have thus obtained the formula

$$f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \frac{1}{r}$$

$$= (-1)^n \frac{(2n)!}{2^n . n!} \cdot \frac{1}{r^{2n+1}} \left(1 - \frac{r^2 \nabla^2}{2.2n-1} + \frac{r^4 \nabla^4}{2.4.2n-1.2n-3} - \ldots\right) f_n(x, y, z)$$

..................(1),

a theorem in differentiation which, as I shall show, is of considerable importance in the theory of spherical harmonics.

If $f_n(x, y, z)$ is a spherical harmonic, the theorem (1) becomes

$$f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} = (-1)^n \frac{(2n)!}{2^n . n!}\, \frac{f_n(x, y, z)}{r^{2n+1}}.$$

This particular case of (1) has been given by Mr. W. D. Niven in his memoir on "Ellipsoidal Harmonics."*

2. The expressions for the ordinary zonal and tesseral harmonics are immediately deducible from (1); putting $f_n(z) = z^n$, we have, if $z = r\mu$,

$$P_n(\mu) = (-1)^n \frac{r^{n+1}}{n!} \frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right)$$

$$= \frac{(2n)!}{2^n . n!\, n!}\left\{\mu^n - \frac{n(n-1)}{2.2n-1}\mu^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.2n-1.2n-3}\mu^{n-4} - \dots\right\}.$$

If we put $\quad f_n(x, y, z) = \left\{(x+\iota y)^m \pm (x-\iota y)^m\right\} z^{n-m}$,

we have

$$\begin{matrix}\cos \\ \sin\end{matrix} m\phi . r_n^m(\mu) = c\left\{\left(\frac{\partial}{\partial x} + \iota \frac{\partial}{\partial y}\right)^m \pm \left(\frac{\partial}{\partial x} - \iota \frac{\partial}{\partial y}\right)^m\right\}\left(\frac{\partial}{\partial z}\right)^{n-m}\frac{1}{r}$$

$$= C \begin{matrix}\cos \\ \sin\end{matrix} m\phi . \sin^m\theta \left\{\mu^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)}\mu^{n-m-2} + \dots\right\},$$

where c and C denote certain constants depending on n and m.

In order to deduce Maxwell's expression for a harmonic having any given poles, we put

$$f_n(x, y, z) = \prod_{s=1}^{s=n}(l_s x + m_s y + n_s z);$$

then, using Maxwell's notation

$$\lambda_s r = l_s x + m_s y + n_s z, \quad \mu_{st} = l_s l_t + m_s m_t + n_s n_t,$$

we find $\quad V^2\Pi(lx+my+nz) = 2\Sigma(\mu'\lambda^{n-2})\, r^{n-2}$;

also $\quad V^4\Pi(lx+my+nz) = 2^2 . 2\Sigma(\mu^2\lambda^{n-4})\, r^{n-4}$,

and generally $V^{2m}\Pi(lx+my+nz) = 2^m\, m!\, \Sigma(\mu^m\lambda^{n-2m})\, r^{n-2m}$.

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* *Phil. Trans.*, Vol. CLXXXII.; see p. 236.

Thus by means of (1) we obtain the expression for Y_n, the surface harmonic which has given poles; this is

$$Y_n = r^{n+1}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial h_1 \partial h_2 \dots \partial h_n}\frac{1}{r}$$

$$= S\left\{(-1)^m\frac{(2n-2m)!}{2^{n-m}n!\,(n-m)!}\Sigma\left(\lambda^{n-2m}\mu^m\right)\right\}.$$

3. It has been remarked by Sylvester* that the determination of the poles of a harmonic $V_n(x, y, z)$ depends on the solution of the equation of degree $2n$, which must be solved in order to solve the simultaneous equations

$$V_n(x, y, z) = 0, \quad x^2+y^2+z^2 = 0;$$

this can be seen by means of (1), for that theorem shows at once that the equations

$$V_n = 0, \qquad x^2+y^2+z^2 = 0$$

are equivalent to $f_n(x, y, z) = 0, \quad x^2+y^2+z^2 = 0,$

where $f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is a differential operator, which, acting on $\frac{1}{r}$, produces V_n; now it must be possible to determine a function U_{n-2}, such that $f_n(x, y, z)+(x^2+y^2+z^2)\,U_{n-2}$ is of the form

$$\prod_{s=1}^{s=n}(l_s x+m_s y+n_s z),$$

where l_s, m_s, n_s are the direction cosines of a pole of V_n, for

$$f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)+\left[\left(\frac{\partial}{\partial x}\right)^2+\left(\frac{\partial}{\partial y}\right)^2+\left(\frac{\partial}{\partial z}\right)^2\right]U_{n-2}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

must be reducible to $\Pi\left(l_s\frac{\partial}{\partial x}+m_s\frac{\partial}{\partial y}+n_s\frac{\partial}{\partial z}\right).$

It follows that the plane $l_s x+m_s y+n_s z = 0$ passes through two of the generating lines of the cone $x^2+y^2+z^2 = 0$ in which that cone is intersected by $f_n(x, y, z) = 0$. Thus a pole (l_s, m_s, n_s) is the pole with respect to the cone $x^2+y^2+z^2 = 0$ of a plane passing through two of the lines of intersection of $V_n = 0$ and $x^2+y^2+z^2 = 0$. A number of systems of poles can thus be determined for a given harmonic V_n, but

* *Phil. Mag.*, 1876.

only one of such system of poles is real, namely that in which the lines in one pair correspond to conjugate imaginary roots of the simultaneous equations $V_n = 0$, $x^2+y^2+z^2 = 0$. Suppose

$$\frac{x}{\alpha_1+\iota\beta_1} = \frac{y}{\alpha_2+\iota\beta_2} = \frac{z}{\alpha_3+\iota\beta_3}$$

gives one set of roots, the conjugate set is given by

$$\frac{x}{\alpha_1-\iota\beta_1} = \frac{y}{\alpha_2-\iota\beta_2} = \frac{z}{\alpha_3-\iota\beta_3},$$

and the corresponding factor $lx+my+nz$ is

$$\begin{vmatrix} x & y & z \\ \alpha_1+\iota\beta_1 & \alpha_2+\iota\beta_2 & \alpha_3+\iota\beta_3 \\ \alpha_1-\iota\beta_1 & \alpha_2-\iota\beta_2 & \alpha_3-\iota\beta_3 \end{vmatrix},$$

which is real. It is obvious that if any other pairs of roots were taken together, the corresponding factor $lx+my+nz$, and therefore the direction cosines (l, m, n) of the corresponding pole, would be imaginary.

4. Let $$f\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial x_0}\frac{\partial}{\partial x}+\frac{\partial}{\partial y_0}\frac{\partial}{\partial y}+\frac{\partial}{\partial z_0}\frac{\partial}{\partial z}\right)^n,$$

where the operators $\frac{\partial}{\partial x_0}, \frac{\partial}{\partial y_0}, \frac{\partial}{\partial z_0}$ are independent of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$; and operate on $(x_0^2+y_0^2+z_0^2)^{-\frac{1}{2}}$. We have then, from (1),

$$\left(\frac{\partial}{\partial x_0}\frac{\partial}{\partial x}+\frac{\partial}{\partial y_0}\frac{\partial}{\partial y}+\frac{\partial}{\partial z_0}\frac{\partial}{\partial z}\right)^n \frac{1}{r}\,\frac{1}{r_0}$$

$$= (-1)^n \frac{(2n)!}{2^n n!}\,\frac{1}{r^{2n+1}}\left\{\left(x\frac{\partial}{\partial x_0}+\dots\right)^n\right.$$

$$\left. -\frac{n(n-1)}{2.2n-1}\,r^2\left(\frac{\partial^2}{\partial x_0^2}+\dots\right)\left(x\frac{\partial}{\partial x_0}+\dots\right)^{n-2}+\dots\right\}\frac{1}{r_0}$$

$$= (-1)^n \frac{(2n)!}{2^n n!}\,\frac{1}{r^{2n+1}}\left(x\frac{\partial}{\partial x_0}+y\frac{\partial}{\partial y_0}+z\frac{\partial}{\partial z_0}\right)^n\frac{1}{r_0}.$$

Now $$P_n\left(\frac{xx_0+yy_0+zz_0}{rr_0}\right) = \frac{(-1)^n r_0^{n+1}}{r^n\, n!}\left(x\frac{\partial}{\partial x_0}+y\frac{\partial}{\partial y_0}+z\frac{\partial}{\partial z_0}\right)^n\frac{1}{r_0};$$

we thus obtain the theorem

$$(2n)!\,P_n\left(\frac{xx_0+yy_0+zz_0}{rr_0}\right) = 2^n r^{n+1} r_0^{n+1}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x_0}+\frac{\partial}{\partial y}\frac{\partial}{\partial y_0}+\frac{\partial}{\partial z}\frac{\partial}{\partial z_0}\right)^n \frac{1}{rr_0},$$

which was first given by Mr. W. D. Niven.*

5. The next application which I shall make of the differentiation theorem is to obtain the expression for an internal ellipsoidal harmonic as a series of spherical harmonics. This important expression was given by Mr. W. D. Niven in the memoir on ellipsoidal harmonics already referred to.

Denoting by G_n an ellipsoidal harmonic of degree n which is of the form

$$\left\{\begin{matrix} & x & yz & \\ 1 & y & zx & xyz \\ & z & xy & \end{matrix}\right\}\Pi\left(\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}-1\right),$$

it is required to express G_n in terms of the spherical harmonic

$$\left\{\begin{matrix} & x & yz & \\ 1 & y & zx & xyz \\ & z & xy & \end{matrix}\right\}\Pi\left(\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}\right),$$

denoted by H_n. At the surface of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

we have, since

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}-1 = -\theta\left(\frac{x^2}{a^2.\,a^2+\theta}+\frac{y^2}{b^2.\,b^2+\theta}+\frac{z^2}{c^2.\,c^2+\theta}\right),$$

$$G_n = (-1)^s\theta_1\theta_2\ldots\theta_s\; H_n\left(\frac{x}{a},\,\frac{y}{b},\,\frac{z}{c}\right)\left\{\begin{matrix} & a & bc & \\ 1 & b & ca & abc \\ & c & ab & \end{matrix}\right\},$$

where s is the number of quadratic factors, and is equal to $\frac{n}{2}$, $\frac{n-1}{2}$, $\frac{n-2}{2}$, or $\frac{n-3}{2}$.

* *Phil. Trans.*, 1879.

Now, since $H(x, y, z)$ is a spherical harmonic, we have

$$H_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r} = \frac{(-1)^n(2n)!}{2^n n!}\,\frac{1}{r^{2n+1}}\,H_n(x, y, z).$$

In this equation, change x, y, z into $\frac{x}{a}$, $\frac{y}{b}$, $\frac{z}{c}$; we then have

$$H_n\left(a\frac{\partial}{\partial x}, b\frac{\partial}{\partial y}, c\frac{\partial}{\partial z}\right)\frac{1}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{2}}}$$

$$= \frac{(-1)^n(2n)!}{2^n n!}\,\frac{H_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{2}(2n+1)}};$$

hence, when $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1,$$

the value of G_n is

$$(-1)^s\,\theta_1, \theta_2 \ldots \theta_s \left\{\begin{matrix} & a & bc & \\ 1 & b & ca & abc \\ & c & ab & \end{matrix}\right\}\frac{2^n n!}{(-1)^n(2n)!}$$

$$\times H_n\left(a\frac{\partial}{\partial x}, b\frac{\partial}{\partial y}, c\frac{\partial}{\partial z}\right)\frac{1}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{2}}}.$$

Again, since $\left(\frac{1}{a^2+\theta}\frac{\partial^2}{\partial x^2}+\frac{1}{b^2+\theta}\frac{\partial^2}{\partial y^2}+\frac{1}{c^2+\theta}\frac{\partial^2}{\partial z^2}\right)\frac{1}{r}$ is unaffected by subtracting $\frac{1}{\theta}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{1}{r}$, it is equal to

$$-\frac{1}{\theta}\left(\frac{a^2}{a^2+\theta}\frac{\partial^2}{\partial x^2}+\frac{b^2}{b^2+\theta}\frac{\partial^2}{\partial y^2}+\frac{c^2}{c^2+\theta}\frac{\partial^2}{\partial z^2}\right)\frac{1}{r};$$

hence we have

$$H_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r}$$

$$= \frac{(-1)^s}{\theta_1\theta_2 \ldots \theta_s}\left\{\begin{matrix} & a^{-1} & b^{-1}c^{-1} & \\ 1 & b^{-1} & c^{-1}a^{-1} & a^{-1}b^{-1}c^{-1} \\ & c^{-1} & a^{-1}b^{-1} & \end{matrix}\right\} H_n\left(a\frac{\partial}{\partial x}, b\frac{\partial}{\partial y}, c\frac{\partial}{\partial z}\right)\frac{1}{r},$$

and on changing x, y, z into $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$, this becomes

$$H_n\left(a\frac{\partial}{\partial x}, b\frac{\partial}{\partial y}, c\frac{\partial}{\partial z}\right)\frac{1}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{2}}}$$

$$= \frac{(-1)^s}{\theta_1\theta_2\dots\theta_s}\left\{1 \quad \begin{matrix} a^{-1} & b^{-1}c^{-1} & \\ b^{-1} & c^{-1}a^{-1} & a^{-1}b^{-1}c^{-1} \\ c^{-1} & a^{-1}b^{-1} & \end{matrix}\right\}$$

$$\times H_n\left(a^2\frac{\partial}{\partial x}, b^2\frac{\partial}{\partial y}, c^2\frac{\partial}{\partial z}\right)\frac{1}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{2}}};$$

hence the value of G_n, when

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

is $$\frac{2^n.n!(-1)^n}{(2n)!}H_n\left(a^2\frac{\partial}{\partial x}, b^2\frac{\partial}{\partial y}, c^2\frac{\partial}{\partial z}\right)\frac{1}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{2}}}.$$

Again, in equation (1), change x, y, z into $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$; we then have

$$f_n\left(a\frac{\partial}{\partial x}, b\frac{\partial}{\partial y}, c\frac{\partial}{\partial z}\right)\frac{1}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{2}}}$$

$$=\frac{(-1)^n(2n)!}{2^n n!}\frac{1}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{2}(2n+1)}}$$

$$\times\left\{1-\frac{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)D^2}{2.2n-1}+\dots\right\}f_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right),$$

where $$D^2=a^2\frac{\partial^2}{\partial x^2}+b^2\frac{\partial^2}{\partial y^2}+c^2\frac{\partial^2}{\partial z^2}.$$

Let $$f_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)=H_n(x, y, z);$$

then $$f_n\left(a\frac{\partial}{\partial x}, b\frac{\partial}{\partial y}, c\frac{\partial}{\partial z}\right)=H_n\left(a^2\frac{\partial}{\partial x}, b^2\frac{\partial}{\partial y}, c^2\frac{\partial}{\partial z}\right);$$

thus we have

$$H_n\left(a^2\frac{\partial}{\partial x},\ b^2\frac{\partial}{\partial y},\ c^2\frac{\partial}{\partial z}\right)\frac{1}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)}$$

$$=\frac{(-1)^n(2n)!}{2^n n!}\,\frac{1}{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{2}(2n+1)}}$$

$$\times\left\{1-\frac{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)D^2}{2.2n-1}+\ldots\right\}H_n(x,y,z);$$

therefore the value of G_n, when

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

is $$\left\{1-\frac{D^2}{2.2n-1}+\frac{D^4}{2.4.2n-1.2n-3}-\ldots\right\}H_n(x,y,z).$$

It follows from this that, for all values of x, y, z,

$$G_n=\left\{1-\frac{D^2}{2.2n-1}+\ldots\right\}H_n+\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)U,$$

where U is some function of degree $n-2$, and lower degrees; now, if a^2, b^2, c^2 be changed into $a^2+\lambda$, $b^2+\lambda$, $c^2+\lambda$, and at the same time each θ is diminished by λ, each of the expressions G_n and $\left(1-\frac{D^2}{2.2n-1}+\ldots\right)H_n$ is unaltered (since

$$D^2=(a^2+\lambda)\frac{\partial^2}{\partial x^2}+(b^2+\lambda)\frac{\partial^2}{\partial y^2}+(c^2+\lambda)\frac{\partial^2}{\partial z^2},$$

when acting on the spherical harmonic H), whereas

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$$

will be altered into $$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}-1,$$

and λ is any arbitrary quantity; thus we must have $U = 0$, and hence

$$G_n = \left(1 - \frac{D^2}{2.2n-1} + \frac{D^4}{2.4.2n-1.2n-3} - \dots\right) H_n,$$

which is Mr. Niven's result. The above proof has the advantage of dealing with the functions of the four types at once.

6. If, as before, $f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ denote a rational homogeneous function of degree n, and if $\phi(u)$ denote any function of u, it is clear that

$$f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi(x^2+y^2+z^2)$$

$$= u_n\phi^{(n)}(x^2+y^2+z^2) + u_{n-2}\phi^{(n-1)}(x^2+y^2+z^2) + u_{n-4}\phi^{(n-3)}(x^2+y^2+z^2) + \dots$$

where $\phi^{(n)}(u) = \frac{d^n}{du^n}\phi(u)$, and u_n, u_{n-2}, u_{n-4} ... denote functions which are independent of the form of ϕ, and depend only upon the form of f_n. We can determine the functions u_n, u_{n-2} ... by giving ϕ any form which is convenient for the purpose.

Let $$\phi(u) = u^{-\frac{1}{2}};$$

then the above equality must be the same as (1); we have

$$\phi^{(n)}(u) = (-1)^n \frac{1.3.5\dots 2n-1}{2^n}\frac{1}{u^{n+\frac{1}{2}}};$$

hence the above equation becomes

$$f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r}$$

$$= (-1)^n\frac{(2n)!}{2^{2n}n!}\frac{1}{r^{2n+1}}\left\{u_n - \frac{2}{2n-1}r^2u_{n-2} + \frac{2^2}{2n-1.2n-3}r^4u_{n-4} - \frac{2^3}{2n-1.2n-3.2n-5}r^6u_{n-6} + \dots\right\};$$

comparing this with (1), we have

$$u_n = 2^n f_n(x, y, z), \quad u_{n-2} = 2^{n-2}\nabla^2 f_n(x, y, z),$$

$$u_{n-4} = \frac{2^{n-2}}{2.4}\nabla^4 f_n(x, y, z), \quad u_{n-6} = \frac{2^{n-3}}{2.4.6}\nabla^6 f_n(x, y, z), \text{ \&c.}$$

We thus obtain the theorem

$$f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi\,(x^2+y^2+z^2)$$

$$= \Big\{ 2^n\phi^{(n)}\,(x^2+y^2+z^2) + \frac{2^{n-2}}{1}\phi^{(n-1)}\,(x^2+y^2+z^2)\,V^2$$

$$+\frac{2^{n-4}}{1.2}\phi^{(n-2)}\,(x^2+y^2+z^2)\,V^4 + \frac{2^{n-6}}{1.2.3}\phi^{(n-3)}\,(x^2+y^2+z^2)\,V^6 + \dots \Big\} f_n\,(x,y,z)$$

$$\dots\dots\dots\dots(2),$$

which is a generalization of (1).

If, in (2), we put

$$f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial x}\right)^n, \text{ and } y=0,\ z=0,$$

we have, as a particular case, the well-known theorem

$$\frac{d^n}{dx^n}\phi\,(x^2) = (2x)^n\,\phi^{(n)}\,(x^2) + \frac{n\,(n-1)}{1}\,(2x)^{n-2}\,\phi^{(n-1)}\,(x^2)$$

$$+\frac{n\,(n-1)(n-2)(n-3)}{1\,.\,2}\,(2x)^{n-4}\,\phi^{(n-2)}\,(x^2) + \dots.$$

7. If, in (2), we put $\phi\,(u) = u^{-\frac{1}{2}s}$,

we obtain the theorem

$$f_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r^s} = (-1)^n\,\frac{s\,(s+2)(s+4)\dots(s+2n-2)}{r^{2n+s}}$$

$$\times \left\{1 - \frac{r^2V^2}{2(2n+s-2)} + \frac{r^4V^4}{2.4(2n+s-2)(2n+s-4)} - \dots\right\} f_n\,(x,y,z)$$

$$\dots\dots\dots\dots(3).$$

This theorem (3) plays the same part in the theory of the functions which Heine* calls spherical harmonics of higher order (*Kugelfunctionen höherer Ordnung*) as the theorem (1) does in the theory of the ordinary spherical harmonics. These generalized functions have been treated of by Prof. Cayley, Mehler, and others. A theory of

* See Heine's *Kugelfunctionen*, Vol. I., pp. 451–464.

poles for these functions may be developed in the same manner as for the ordinary case $s = 1$. The tesseral function is seen to be

$$T_n^m(s,\mu) = A\,(1-\mu^2)^{\frac{1}{2}m}\left\{\mu^{n-m} - \frac{(n-m)(n-m-1)}{2\,.\,2n+s-2}\,\mu^{n-m-2}\right.$$
$$\left. + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2\,.\,4\,(2n+s-2)(2n+s-4)}\,\mu^{n-m-4} - \dots\right\},$$

where A is a constant; in Heine's notation this function is $P_m^n(s+1,\mu)$.

I find, by a process exactly similar to that in Section 2, the following expression for the generalized harmonic with given poles:

$$Y_{n,s} = r^{n+s}\frac{(-1)^n}{n!}\,\frac{\partial^n}{\partial h_1\,\partial h_2\dots\partial h_n}\,\frac{1}{r^s}$$
$$= S\left\{\frac{(-1)^m}{n!}\,\frac{s\,(s+2)\dots(s+2n-2)}{(2n+s-2)(2n+s-4)\dots(2n+s-2m)}\,\Sigma\,(\lambda^{n-2m}\mu^m)\right\}.$$

In the case $s = -1$, these functions are of use in applications to physical problems; they have been considered by Messrs. Butcher* and Sampson.†

8. The theorems (1), (2), and (3) can be extended to the case of p variables $x_1, x_2, x_3 \dots x_p$. In this case I find, by processes exactly similar to those used in the case $p = 3$,

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\dots\frac{\partial}{\partial x_p}\right)\frac{1}{r^{p-2}}$$
$$= (-1)^n\,(p-2)\,p\,(p+2)\dots(p+2n-4)\,\frac{1}{r^{2n+p-2}}$$
$$\times\left\{1 - \frac{r^2V^2}{2\,(2n+p-4)} + \frac{r^4V^4}{2.4(2n+p-4)(2n+p-6)} - \dots\right\}f_n(x_1, x_2 \dots x_p)$$
$$\dots\dots\dots\dots\dots\dots(4),$$

where $\quad r^2 = x_1^2 + x_2^2 + \dots + x_n^2, \quad V^2 = \left(\frac{\partial}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial}{\partial x_p}\right)^2.$

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* *Lond. Math. Soc. Proc.*, Vol. VIII.
† *Phil. Trans.*, 1891.

This is the generalization of (1), $\frac{1}{r^{p-2}}$ being the hyperpotential function which satisfies the equation

$$\nabla^2 \frac{1}{r^{p-2}} = 0.$$

Also
$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \dots \frac{\partial}{\partial x_p}\right)\phi\,(x_1^2+x_2^2+\dots+x_p^2)$$
$$= \left\{2^n\phi^{(n)}(x_1^2+\dots) + \frac{2^{n-1}}{2}\phi^{(n-1)}(x_1^2+\dots)\,\nabla^2 + \frac{2^{n-2}}{2.4}\phi^{(n-2)}(x_1^2+\dots)\,\nabla^4+\dots\right\} f_n(x_1, x_2 \dots x_p)\dots(5).$$

This is the extension of (2) to any number of variables.

The theorem (4) may be used, as in the case $p=3$, to find a general expression for the hyper-spherical harmonic which has given poles, but, as the method contains nothing new, I shall not proceed further with this extension. The theorem (5) is the most general differentiation theorem of those which I have given, as it contains all the others as particular cases; it may, I think, be regarded as fundamental in the theory of hyper-spherical or spherical harmonics of all orders.

Notes on Determinants. By J. E. Campbell.

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[In accordance with the late Professor Smith's notation, a determinant of the p^{th} class may be written

$$|\, a_{ijk} \dots \,|\,.$$

The fact that a determinant of the second class (an ordinary determinant) is not altered if the vertical columns be written horizontally, is expressed by the identity

$$|\, a_{ij} \,| = |\, a_{ji} \,|\,.$$

For determinants of higher class it is known that any of the suffixes can be interchanged except the first; and, if the class be even,

the first suffix can also be interchanged with any other, but for determinants of odd class this is not true.

By considering a cubic determinant as an ordinary determinant in alternate numbers, I have tried to explain this essential distinction between determinants of odd and even classes.

If the element $\quad a_{pqr\ldots} = -a_{qpr\ldots},\;$ and $\; a_{ppr} = 0,$

the determinant is called skew symmetrical.

It is easily seen that skew symmetrical determinants of even class and odd degree vanish identically. This is analogous to the well-known theorem in ordinary determinants; but there is no corresponding analogue to the theorem that skew symmetrical determinants of the second class and even degree are perfect squares.

The reasoning which establishes these propositions does not apply to skew symmetrical determinants of odd class.

By a different method it is shown that they vanish identically whether the class be even or odd.

It is next shown that, if we form any determinant of even class $2p$, from $2p$ ordinary determinants, in a manner analogous to that in the rule for the multiplication of two ordinary determinants, the determinant so formed is the product of the $2p$ determinants; and if any determinant of odd class $2p+1$ is formed from $2p+1$ ordinary determinants, the determinant so formed is the product of the last $2p$ of these ordinary determinants into the first, taken with all its signs positive.

A somewhat similar result is shown to hold for determinants of alternate numbers.

As an application, let

$$Z = \frac{a_1}{(x-a_1)(y-\beta_1)} + \ldots + \frac{a_n}{(x-a_n)(y-\beta_n)},$$

and let (p, q) denote $\dfrac{d^{p+q}z}{p!\,q!\,dx^p dy^q}$.

By multiplying the arrays

$$\begin{Vmatrix} \dfrac{a_1}{x-a_1} & \cdots\cdots \\ \dfrac{a_1}{(x-a_1)^2} & \cdots\cdots \end{Vmatrix}, \qquad \begin{Vmatrix} \dfrac{1}{y-\beta_1} & \cdots\cdots \\ \dfrac{1}{(y-\beta_1)^2} & \cdots\cdots \end{Vmatrix},$$

we get $\begin{vmatrix} (0,0), & (0,1) \\ (1,0), & (1,1) \end{vmatrix} = \Sigma \dfrac{a_p a_q (a_p - a_q)(\beta_p - \beta_q)}{(x-a_p)^2 (x-\beta_p)^2 (x-a_q)^2 (x-\beta^q)^2}.$

Suppose now $n = 1$; we get that the primitive of

$$\begin{vmatrix} (0,0), & (0,1) \\ (1,0), & (1,1) \end{vmatrix}$$

is
$$Z = \frac{a_1}{(x-a_1)(y-\beta_1)}.$$

Similarly, by multiplying

$$\begin{Vmatrix} \dfrac{a_1}{(x-a_1)} & \cdots\cdots \\ \dfrac{a_1}{(x-a_1)^2} & \cdots\cdots \\ \dfrac{a_1}{(x-a_1)^3} & \cdots\cdots \end{Vmatrix}, \qquad \begin{Vmatrix} \dfrac{1}{y-\beta_1} & \cdots\cdots \\ \dfrac{1}{(y-\beta_1)^2} & \cdots\cdots \\ \dfrac{1}{(y-\beta_1)^3} & \cdots\cdots \end{Vmatrix},$$

we get that the primitive of

$$\begin{vmatrix} (0,0), & (0,1), & (0,2) \\ (1,0), & (1,1), & (1,2) \\ (2,0), & (2,1), & (2,2) \end{vmatrix} = 0$$

is
$$Z = \frac{a_1}{(x-a_1)(y-\beta_1)} + \frac{a_2}{(x-a_2)(y-\beta_2)}.$$

Similar primitives are obtained for differential equations which are in the form of determinants of higher class.

A further application is obtained by taking powers of different invariantive symbols, of which (123) is the simplest for the ternary quantic.

The resulting invariants are seen to be determinants of some even class.]

In accordance with Prof. Sylvester's umbral notation, a cubic determinant is written

$$\begin{vmatrix} 1, & 1, & 1 \\ 2, & 2, & 2 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ n, & n, & n \end{vmatrix},$$

and denotes the sum of all possible products

$$a_{111} \cdot a_{222} \cdots a_{nnn},$$

obtained by giving the terms in the second and third columns every possible permutation, and changing the sign with each permutation.

Similarly determinants of the fourth class may be written

$$\left| \begin{array}{cccc} 1, & 1, & 1, & 1 \\ 2, & 2, & 2, & 2 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ n, & n, & n, & n \end{array} \right|;$$

and so on for determinants of higher class.

Shorter ways of writing these would be

$$| a_{ijk} |, \quad | a_{ijkl} |, \ \ldots\ldots$$

The fact that a determinant of the second class (an ordinary determinant) is not altered if the vertical rows be written horizontally, is expressed by the identity

$$| a_{ij} | = | a_{ji} | .$$

It is shown by Prof. Lloyd Tanner (*Proc. Lond. Math. Soc.*, Vol. x.), that

$$| a_{ijkl} | = | a_{ikjl} |, \quad \text{and} \quad | a_{ijk} | = | a_{ikj} |;$$

and generally that any of the suffixes can be interchanged except the first.

For determinants of even class, the first suffix can also be interchanged with any other, but this is not true for determinants of odd class; thus

$$| a_{ijkl} | = | a_{jikl} |;$$

but $| a_{ijk} |$ is not equal to $| a_{jik} |$.

To consider more closely the case of determinants of odd class, take a cubic determinant, or its equivalent, an ordinary determinant of alternate numbers (Scott, *Proc. Lond. Math. Soc.*, Vol. xi.), $| a_{ij} |$.

I shall show that $$| a_{ij} | = \{ a_{ji} \},$$

where $\{ \ \}$ denotes a determinant with all the signs taken positive.

Consider any term in $| a_{ij} |$, $a_{13} . a_{21} . a_{32}$; to this there will be a complementary term of the same sign

$$a_{12} . a_{23} . a_{31}.$$

Corresponding to these, in the signless determinant $\{a_{ji}\}$, we have

$$a_{31} . a_{12} . a_{23},$$

and its complementary $\qquad a_{21} . a_{32} . a_{13}.$

To bring these to the forms $a_{12} . a_{23} . a_{31}$, and $a_{13} . a_{21} . a_{32}$, respectively, the alternate numbers must be inverted precisely the same number of times that the symbols were to deduce 13 . 21 . 32 from 11.22..33. The signs therefore of $a_{12} . a_{23} . a_{31}$ and $a_{13} . a_{21} . a_{32}$, in $\{a_{ji}\}$, will be the same as the signs of the corresponding terms in $| a_{ij} |$; so that we have

$$| a_{ij} | = \{ a_{ji} \}.$$

Just as it was shown that cubic determinants were equivalent to ordinary determinants of alternate numbers, it may be shown that determinants of odd class generally are equivalent to ordinary determinants of odd products of sets of alternate numbers, and even-classed determinants to ordinary determinants of even products of alternate numbers. The reasoning applied to cubic determinants will then explain the essential distinction between the odd and even classes.

If the element $\quad a_{ijk...} = - a_{jik...} \quad$ and $\quad a_{iik} = 0,$

the determinant is called skew symmetrical.

It is then easily shown that a skew symmetrical determinant of even class and odd degree vanishes.

For $\qquad a_{ijkl} = - a_{jikl},$

and the degree is odd; therefore

$$| a_{ijkl} | = - | a_{jikl} |;$$

and therefore $\qquad | a_{ijkl} | = 0,$

since $| a_{ijkl} |$ also equals $| a_{jikl} |$.

This theorem is analogous to the corresponding one in ordinary determinants; but we have no theorem for higher determinants analogous to the one that skew symmetrical determinants of even degree are perfect squares.

This reasoning would not apply to determinants of odd class, since we have not the fundamental formula

$$| a_{ijk} | = | a_{jik} | .$$

I shall show, however, by another method, that skew symmetrical determinants of odd class vanish, whether their degree be even or odd.

Consider a cubic determinant, or its equivalent, an ordinary determinant of alternate numbers.

Let $a_{11} . a_{22} . a_{33} . a_{44} . a_{55}$ be the leading term. Every other term is derived from this by cyclical permutations of groups of the elements.

Take any permutation 245.31. This generates the term $a_{13} . a_{25} . a_{31} . a_{42} . a_{54}$, and from this, by the permutation 452, we get another term $a_{13} . a_{24} . a_{31} . a_{45} . a_{52}$. Since the number of terms in the cycle is odd, we can write this in the form $a_{13} . a_{52} . a_{31} . a_{24} . a_{45}$ without altering the sign. Add now these terms $a_{13} . a_{52} . a_{31} . a_{24} . a_{45}$ and $a_{13} . a_{25} . a_{31} . a_{42} . a_{54}$; they cancel one another, since

$$a_{25}+a_{52} = 0, \quad a_{24}+a_{42} = 0, \quad a_{45}+a_{54} = 0.$$

Similarly, we see that every other term which appears will have a term cancelling it; unless all the cycles of the group are mere inversions; and these also disappear for

$$a_{pq} . a_{qp} = -(a_{pq})^2 = 0,$$

since a_{pq} is an alternate number. The only term now left in $a_{11} . a_{22} . a_{33} . a_{44} . a_{55}$, and each factor of this is zero.

Similarly, the theorem may be proved for any determinant of odd class; the essential part of the proof consisting in noticing that the term obtained by a right-handed cyclical permutation of a group (exceeding two) is of the same sign as the term obtained by the corresponding left-handed permutation.

If the determinant had been only half skew, that is, if

$$a_{ijk} = - a_{jik} ,$$

but a_{iik} not zero, the determinant would have reduced to $a_{11} . a_{22} \dots a_{nn}$ in alternate numbers; that is, to the ordinary determinant

$$\begin{vmatrix} a_{111}, & a_{112}, & \dots\dots & a_{11n} \\ a_{221}, & a_{222}, & \dots\dots & a_{22n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{nn1}, & a_{nn2}, & \dots\dots & a_{nnn} \end{vmatrix} .$$

If a cubic determinant be formed with the elements

$$(ijk) = \Sigma_{p=1}^{p=n} a_{ip} \,.\, b_{jp} \,.\, c_{kp},$$

it is equal to the product of the ordinary determinants $| b_{ik} |$, $| c_{ik} |$, and the signless determinant $\{a_{ik}\}$.

The cubic determinant is

$$\Pi_{i=1}^{i=n} \Sigma_{j,k=1,1}^{j,k=n,n} (ijk)\, \epsilon_j\, e_k,$$

where $\epsilon_1, \epsilon_2 \dots e_1, e_2 \dots$ are two independent sets of alternate units (Scott, *Theory of Determinants*, VII., 3).

$$\Sigma_{j=1}^{j=n} (ijk)\, \epsilon_j e_k = \Sigma_{p=1}^{p=n} a_{ip} c_{kp} \beta_p\, e_k,$$

where β_p is the alternate number $b_{1p}\epsilon_1 + b_{2p}\epsilon_2 + \dots + b_{np}\epsilon_n$.

$$\Sigma_{j,k}^{j,k} (ijk)\, \epsilon_j e_k = \Sigma_{p=1}^{p=n} a_{ip} \beta_p \gamma_p,$$

where γ_p is the alternate number $c_{1p} e_1 + c_{2p} e_2 + \dots + c_{np} e_n$.

Therefore the cubic determinant

$$= \Pi_{i=1}^{i=n} (a_{i1}\beta_1\gamma_1 + a_{i2}\beta_2\gamma_2 + \dots + a_{in}\beta_n\gamma_n).$$

Now $\beta\gamma$, being the product of two alternate numbers, is a concurrent number, so that the determinant reduces to

$$\{a_{ik}\}\, \beta_1\gamma_1\, \beta_2\gamma_2 \dots \beta_n\gamma_n,$$

and $$\beta_1\beta_2 \dots \beta_n = | b_{ik} |,$$

and $$\gamma_1\gamma_2 \dots \gamma_n = | c_{ik} |;$$

so that, finally, the cubic determinant

$$= \{ a_{ik} \} \times | b_{ik} | \times | c_{ik} | .$$

If a determinant of the fourth class be formed with the elements

$$(ijkl) = \Sigma_{p=1}^{p=n} a_{ip}\, b_{jp}\, c_{kp}\, d_{lp},$$

it may be shown by a similar method that it is the product of four ordinary determinants

$$| a_{ik} | \times | b_{ik} | \times | c_{ik} | \times | d_{ik} |,$$

for it is reduced to the product

$$\Pi_{i=1}^{i=n} (a_{i1}\beta_1\gamma_1\delta_1 + a_{i2}\beta_2\gamma_2\delta_2 + \dots + a_{in}\beta_n\gamma_n\delta_n),$$

and $\beta \,.\, \gamma \,.\, \delta$, being the product of three alternate numbers, is itself an alternate number.

And so generally, if the class p of the determinant formed by this method is odd, it is the product of $p-1$ ordinary determinants and a signless determinant $\{a_{ik}\}$, where a is the letter of the first determinant of the auxiliary system ; and, if the class p is even, the determinant is the product of the p auxiliary determinants.

If the number of rows and columns in the auxiliary matrices be not equal, similar reasoning leads to analogous results to those in the ordinary multiplication of matrices.

A rule is given by the late Mr. Spottiswoode (*Proc. Lond. Math. Soc.*, February, 1876) for the multiplication of two determinants of alternate numbers. A more convenient rule (with a view to this extension) would be:—

The product of two such determinants $= \pm$ the corresponding compound determinant taken with all the signs positive.

The general rule for multiplication is then the same as for determinants of ordinary numbers, if the resulting determinant of the p^{th} class is taken with all its signs positive.

This may easily be deduced from the former method by employing sets of concurrent units instead of alternate ones.

$$\text{Let } Z = \frac{a_1}{(x-\alpha_1)(y-\beta_1)} + \frac{a_2}{(x-\alpha_2)(y-\beta_2)} + \ldots + \frac{a_n}{(x-\alpha_n)(y-\beta_n)},$$

and let (pq) denote $\dfrac{\partial^{p+q} z}{p!\,q!\,\partial x^p \partial y^q}$. Then

$$(0,\,0) = \frac{a_1}{(x-\alpha_1)(y-\beta_1)} + \ldots,$$

$$-(0,\,1) = \frac{a_1}{(x-\alpha_1)(y-\beta_1)^2} + \ldots,$$

$$(-1)^{p+q}\,(p,\,q) = \frac{a_1}{(x-\alpha_1)^{p+1}\,(y-\beta_1)^{q+1}} + \ldots.$$

By multiplying the rectangular arrays

$$\left\| \begin{array}{cccc} \dfrac{a_1}{x-\alpha_1}, & \dfrac{a_2}{x-\alpha_2}, & \ldots \\[2ex] \dfrac{a_1}{(x-\alpha_1)^2}, & \dfrac{a_2}{(x-\alpha_2)^2}, & \ldots \\[2ex] \ldots & \ldots & \ldots & \ldots \end{array} \right\| \times \left| \begin{array}{cccc} \dfrac{1}{y-\beta_1}, & \dfrac{1}{y-\beta_2}, & \ldots \\[2ex] \dfrac{1}{(y-\beta_1)^2}, & \dfrac{1}{(y-\beta_2)^2}, & \ldots \\[2ex] \ldots & \ldots & \ldots & \ldots \end{array} \right|,$$

we have the series

$$\begin{vmatrix} (0,0), & (0,1) \\ (1,0), & (1,1) \end{vmatrix} \equiv \Sigma \frac{a_p a_q (\alpha_p - \alpha_q)(\beta_p - \beta_q)}{(x-\alpha_p)^2 (y-\beta_p)^2 (x-\alpha_q)^2 (y-\beta_q)^2},$$

$$\begin{vmatrix} (0,0), & (0,1), & (0,2) \\ (1,0), & (1,1), & (1,2) \\ (2,0), & (2,1), & (2,2) \end{vmatrix}$$

$$\equiv \Sigma \frac{a_p a_q a_r \rho^{\frac{1}{2}} (\alpha_p \alpha_q \alpha_r)\, \rho^{\frac{1}{2}} (\beta_p \beta_q \beta_r)}{(x-\alpha_p)^3 (y-\beta_p)^3 (x-\alpha_q)^3 (y-\beta_q)^3 (x-\alpha_r)^3 (y-\beta_r)^3},$$

and similar functions, where the summation extends to all values of p, q, r selected from the numbers 1, 2, 3 ... n.

Let $n = 2$; we have then

$$\begin{vmatrix} (0,0), & (0,1), & (0,2) \\ (1,0), & (1,1), & (1,2) \\ (2,0), & (2,1), & (2,2) \end{vmatrix} = 0,$$

so that the primitive of this equation is

$$Z = \frac{a_1}{(x-\alpha_1)(y-\beta_1)} + \frac{a_2}{(x-\alpha_2)(y-\beta_2)},$$

where a_1, a_2, α_1, β_1, α_2, β_2, are arbitrary constants.

This was suggested by a paper of Mr. Forsyth's in the *Messenger of Mathematics* (February, 1888), in which, by a very simple method, similar equations were obtained for the case of one independent variable.

So, by multiplying together four matrices, we should obtain

$$\begin{vmatrix} 0, & 0, & 0, & 0 \\ 1, & 1, & 1, & 1 \end{vmatrix}$$

$$\equiv \Sigma \frac{a_p a_q (\alpha_p - \alpha_q)(\beta_p - \beta_q)(\gamma_p - \gamma_q)(\delta_p - \delta_q)}{(x_1-\alpha_p)^2 (x_2-\beta_p)^2 (x_3-\gamma_p)^2 (x_4-\delta_p)^2 (x_1-\alpha_q)^2 (x_2-\beta_q)^2 (x_3-\gamma_q)^2 \times (x_4-\delta_q)^2},$$

$$\begin{vmatrix} 0, & 0, & 0, & 0 \\ 1, & 1, & 1, & 1 \\ 2, & 2, & 2, & 2 \end{vmatrix}$$

$$\equiv \Sigma \frac{a_p a_q a_r \rho^{\frac{1}{2}}(\alpha_p, \alpha_q, \alpha_r)\, \rho^{\frac{1}{2}}(\beta_p, \beta_q, \beta_r)\, \rho^{\frac{1}{2}}(\gamma_p, \gamma_q, \gamma_r)\, \rho^{\frac{1}{2}}(\delta_p, \delta_q, \delta_r)}{(x_1-\alpha_p)^3(x_2-\beta_p)^3(x_3-\gamma_p)^3(x_4-\delta_p)^3(x_1-\alpha_q)^3(x_2-\beta_q)^3(x_3-\gamma_q)^3 \times (x_4-\delta_q)^3(x_1-\alpha_r)^3(x_2-\beta_r)^3(x_3-\gamma_r)^3(x_4-\delta_r)^3},$$

where $\rho^{\frac{1}{2}}(\alpha_p, \alpha_q, \alpha_r)$ denotes the product

$$(\alpha_p-\alpha_q)(\alpha_p-\alpha_r)(\alpha_q-\alpha_r),$$

and where the expressions on the left denote determinants of the fourth class, and 2nd, 3rd, ... degrees, whose elements

$$(p, q, r, s) = \frac{\partial^{p+q+r+s} Z}{p!\,q!\,r!\,s!\,\partial x_1^p \,.\, \partial x_2^q \,.\, \partial x_3^r \,.\, \partial x_4^s}.$$

Thus we get: the primitive of

$$\begin{vmatrix} 0, & 0, & 0, & 0 \\ 1, & 1, & 1, & 1 \end{vmatrix} = 0$$

is
$$Z = \frac{a_1}{(x_1-\alpha_1)(x_2-\beta_1)(x_3-\gamma_1)(x_4-\delta_1)},$$

the primitive of
$$\begin{vmatrix} 0, & 0, & 0, & 0 \\ 1, & 1, & 1, & 1 \\ 2, & 2, & 2, & 2 \end{vmatrix}$$

is

$$Z = \frac{a_1}{(x_1-\alpha_1)(x_2-\beta_1)(x_3-\gamma_1)(x_4-\delta_1)} + \frac{a_2}{(x_1-\alpha_2)(x_2-\beta_2)(x_3-\gamma_2)(x_4-\delta_2)},$$

and so on.

Similar reasoning would apply to determinants of any even class. The results for determinants of odd class are not quite so symmetrical.

In Mr. Forsyth's paper he shows that his functions are homographic invariants; but this is easily seen (from the form of the primitive) not to be true when the differential equations are partial, as here.

$$(12)^n, \quad \{(23)(31)(12)\}^n, \quad \{(23)(31)(12)(14)(24)(34)\}^n, \ \ldots$$

are invariants of the even binary quantic.

It is shown, amongst other matter, in the papers of Messrs. Cayley and Sylvester in the *Cambridge and Dublin Mathematical Journal*, 1852, that these may be expressed as commutants. The following is, perhaps, a slightly different way of looking at the matter.

The binary quantic being

$$(\alpha_1 x_1 + \alpha_2 x_2)^n + (\beta_1 x_1 + \beta_2 x_2)^n + \dots,$$

we can form, from

$$\left\| \begin{matrix} \alpha_1, & \beta_1, & \dots\dots \\ \alpha_2, & \beta_2, & \dots\dots \end{matrix} \right\|^n,$$

a determinant of the n^{th} class and second degree, which will be equal to the sum

$$\Sigma \begin{pmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \end{pmatrix}^n.$$

Now each of the expressions $\begin{vmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \end{vmatrix}$ is unaltered by linear transformation, and the determinant of the n^{th} class is at once seen to be a rational function of the coefficients of the binary quantic; it is therefore an invariant.

So, from

$$\left\| \begin{matrix} \alpha_1^2, & \beta_1^2, & \dots \\ \alpha_1\alpha_2, & \beta_1\beta_2, & \dots \\ \alpha_2^2, & \beta_2^2, & \dots \end{matrix} \right\|^n$$

we can generate a determinant of the n^{th} class and third degree,

$$\equiv \Sigma \begin{vmatrix} \alpha_1^2, & \beta_1^2, & \gamma_1^2 \\ \alpha_1\alpha_2, & \beta_1\beta_2, & \gamma_1\gamma_2 \\ \alpha_2^2, & \beta_2^2, & \gamma_2^2 \end{vmatrix}^n$$

$$\equiv \Sigma \begin{vmatrix} \alpha_1, & \beta_1 \\ \alpha_2, & \beta_2 \end{vmatrix}^n \times \begin{vmatrix} \beta_1, & \gamma_1 \\ \beta_2, & \gamma_2 \end{vmatrix}^n \times \begin{vmatrix} \gamma_1, & \alpha_1 \\ \gamma_2, & \alpha_2 \end{vmatrix}^n;$$

and, as each of these expressions is unaltered by linear transformation, the determinant will be an invariant of

$$(\alpha_1 x_1 + \alpha_2 x_2)^{2n} + (\beta_1 x_1 + \beta_2 x_2)^{2n} + \dots;$$

and so generally for binary quantics.

For ternary quantics, $(123)^n$ is an invariantive symbol if n be even.

Another such symbol is derived from the condition that six points lie on a conic; it is

$$\left|\begin{array}{ccc} \frac{d^2}{dx_1^2}, & \frac{d^2}{dy_1^2}, & \ldots\ldots \\ \frac{d^2}{dx_1 dx_2}, & \frac{d^2}{dy_1 dy_2}, & \ldots\ldots \\ \frac{d^2}{dx_1 dx_3}, & \frac{d^2}{dy_1 dy_3}, & \ldots\ldots \\ \frac{d^2}{dx_2^2}, & \frac{d^2}{dy_2^2}, & \ldots\ldots \\ \frac{d^2}{dx_2 dx_3}, & \frac{d^2}{dy_2 dy_3}, & \ldots\ldots \\ \frac{d^2}{dx_3^2}, & \frac{d^2}{dy_3^2}, & \ldots\ldots \end{array}\right|^n.$$

It might more shortly be written

$$(1,\ 2,\ 3,\ 4,\ 5,\ 6)_2^n \quad \text{or} \quad (6)_2^n.$$

Similarly, from the condition that ten points should lie on a cubic, is derived the operator $(10)_3^n$.

All such invariants can be expressed as determinants of the n^{th} class.

To see this we have only to write the ternary quantic

$$(a_1x_1+a_2x_2+a_3x_3)^n+(\beta_1x_1+\beta_2x_2+\beta_3x_3)^n+\ldots,$$

and for $\frac{d^2}{dx_1^2}$ write a_1^2, for $\frac{d^2}{dx_1dx_2}$ write a_1a_2, and so on.

Then the same method as applied to the binary quantic applies to the ternary.

Thus $(1, 2, 3)^2$, for the conic $\equiv (a_1, \beta_2, \gamma_3)^2$, which vanishes if the conic can be thrown into the form of the sum of two squares.

$(1, 2, 3)^4$, applied to the quartic, gives the invariant A (Salmon, *Higher Plane Curves*, Art. 293).

$(1, 2, 3, 4, 5, 6)_2^2$ applied to the quartic, gives the invariant B.

It is an ordinary determinant, and it vanishes if the six lines $a=0$, $\beta=0$, $\gamma=0$, $\delta=0$, $\epsilon=0$, $\theta=0$, touch a conic, or (as a particular case) if the quartic can be expressed as the sum of five fourth powers.

$(10)^2_3$, applied to a curve of the sixth degree, gives an invariant which vanishes if the sides of what might be called the decagon of reference touch a curve of the third class.

So $(15)^2_4$, applied to a curve of the eighth degree, gives an invariant whose vanishing is the condition that the sides of the quindecagon of reference touch a curve of the fourth class.

It would not be so easy to interpret geometrically the invariants which are in the form of determinants of higher class.

A similar method will apply to equations in any number of variables.

Thursday, January 12th, 1893.

A. B. KEMPE, Esq., F.R.S., President, in the Chair.

Mr. Pat Doyle, C.E., Editor of *Indian Engineering*, Calcutta, was elected a member, and Mr. Greenstreet was admitted into the Society.

The following communications were made:—

On the Application of Clifford's Graphs to Ordinary Binary Quantics—second part, Seminvariants: the President (Prof. Elliott in the chair).

On the Evaluation of a certain Surface-Integral, and its Application to the Expansion of the Potential of Ellipsoids in Series: Dr. Hobson.

Mr. Love made a brief statement on "The Vibrations of an Elastic Circular Ring."

A cabinet likeness of Lt.-Col. Campbell was received and placed in the Society's album.

The following presents were made to the Library:—

"Beiblätter zu den Annalen der Physik und Chemie," Band xvi., Stück 11.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. xi., No. 2, 1892; Coimbra.

"Principles of the Algebra of Physics," by A. Macfarlane; pamphlet, 8vo, Salem, Mass., 1891.

"Bulletin of the New York Mathematical Society," Vol. ii., No. 3.

"Bulletin de la Société Mathématique de France," Tome xx., No. 6; Paris, 1892.

"Bulletin des Sciences Mathématiques," Tome xvi.; November, 1892.

"Transactions of the Texas Academy of Science," Vol. i., No. 1; November, 1892.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. i., Fasc. 10, 11, 2° Sem.

"Educational Times," January, 1893.

"Journal für die reine und angewandte Mathematik," Bd. cx., Heft iv.

"Annals of Mathematics," Vol. vii., No. 1; Virginia, November, 1892.

"Indian Engineering," Vol. xii., Nos. 21–25.

"The Stresses in Statistically Intermediate Structures"; Reprint from "Indian Engineering."

"Memoirs of the National Academy of Sciences," Vol. v.; 4to, Washington, 1891.

"Cayley's Collected Mathematical Papers," Vol. v.; 4to, 1892. Two copies.

On the Evaluation of a certain Surface-Integral, and its application to the Expansion, in Series, of the Potential of Ellipsoids. By E. W. Hobson. Received and read January 12th, 1893.

If V be any function of x, y, z, the coordinates of a point, the function being finite and continuous throughout a sphere of radius R whose centre is the origin, it is known that

$$\iint V dS = 4\pi R^2 \sum_{n=0}^{n=\infty} \frac{R^{2n}}{(2n+1)!} \nabla^{2n} V,$$

the integration being taken over the whole surface of the sphere, and $\nabla^{2n} V$ having its value at the origin; ∇^2 denotes Laplace's operator. This theorem has been applied by Mr. W. D. Niven* to the evaluation of a number of important definite integrals involving spherical harmonics, and to the development, in series, of the potentials of a uniform solid ellipsoid and of a homœoid.

I propose here to investigate a more general surface-integral theorem which includes the above, and which also furnishes a proof

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* *Phil. Trans.*, 1879.

and an extension of an important surface-integral theorem due to Maxwell. The theorem is then applied to the determination of the expression for an external ellipsoidal harmonic in a series of spherical harmonics. I have then shown how to obtain expansions of the potentials of ellipsoidal shells, solid ellipsoids, and elliptic discs of variable density, the law of force being any given function of the distance.

The formulae given by most writers on the subject of the attraction of ellipsoids, express the potentials in the form of definite integrals; such formulae have been given by Dr. Ferrers,* and recently in a very elegant form by Mr. Dyson.† The formulae given in the present communication are of such a character that approximate values of the potentials may be obtained by taking as many terms of the series as may be necessary, whereas the definite integral formulae do not lend themselves readily to such approximation.

1. It is known‡ that the expansion of $e^{\iota r \cos\theta}$ in a series of zonal harmonics $P_n(\cos\theta)$ is given by

$$e^{\iota r \cos\theta} = \sum_{n=0}^{n=\infty} \frac{(\iota r)^n}{3.5.7\ldots(2n-1)} \left\{ 1 - \frac{r^2}{2.2n+3} + \frac{r^4}{2.4.2n+3.2n+5} - \ldots \right\}$$
$$\times P_n(\cos\theta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This expansion may be conveniently obtained as follows:—

The differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + V = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

is satisfied by the expressions

$$r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(r)\, P_n(\cos\theta), \quad r^{-\frac{1}{2}} Y_{n+\frac{1}{2}}(r)\, P_n(\cos\theta),$$

where $J_{n+\frac{1}{2}}(r)$, $Y_{n+\frac{1}{2}}(r)$ denote the two Bessel's functions of order $n+\frac{1}{2}$; the functions $r^{-\frac{1}{2}}J_{n+\frac{1}{2}}$ and $r^{-\frac{1}{2}}Y_{n+\frac{1}{2}}$ are of the forms

$$Ar^n \frac{d^n}{d(r^2)^n} \frac{\sin r}{r}, \quad Br^{-n-1} \frac{d^n}{d(r^2)^n} \frac{\cos r}{r},$$

respectively, A and B being constants. Now (2) is satisfied by

* *Quarterly Journal*, Vol. xiv.
† *Quarterly Journal*, Vol. xxv.
‡ See Heine's *Kugelfunctionen*, Vol. i., p. 82.

$V = e^{iz} = e^{ir\cos\theta}$; thus, if $e^{ir\cos\theta}$ be expanded in a series of the harmonics $P_n(\cos\theta)$, we should expect the general term to be

$$\left\{ A_n r^n \frac{d^n}{d(r^2)^n}\,\frac{\sin r}{r} + B_n r^{-n-1} \frac{d^n}{d(r^2)^n}\,\frac{\cos r}{r} \right\} P_n(\cos\theta).$$

It is clear that we must have $B_n = 0$, as the expression must be finite, when $r = 0$; thus

$$e^{ir\cos\theta} = \Sigma A_n r^n \frac{d^n}{d(r^2)^n}\,\frac{\sin r}{r} P_n(\cos\theta),$$

or

$$e^{ir\cos\theta}$$

$$= \Sigma \frac{(-1)^n n!}{(2n+1)!} A_n r^n \left\{ 1 - \frac{r^2}{2.2n+3} + \frac{r^4}{2.4.2n+3.2n+5} - \dots \right\} P_n(\cos\theta).$$

Equating the coefficients of the term $r^n\cos^n\theta$ on both sides of the equation, we have

$$\frac{\iota^n}{n!} = \frac{(-1)^n n!}{(2n+1)!}\,\frac{(2n)!}{2^n n!\, n!} A_n,$$

or

$$\frac{(-1)^n n!}{(2n+1)!} A_n = \frac{\iota^n}{3.5\dots(2n-1)},$$

and thus the expansion (1) is proved. In (1), change r into $-\iota\rho$; we thus obtain the expansion

$$e^{\rho\cos\theta} = \sum_{n=0}^{n=\infty} (2n+1) \frac{\rho^n}{3.5\dots 2n+1}$$

$$\times \left\{ 1 + \frac{\rho^2}{2.2n+3} + \frac{\rho^4}{2.4.2n+3.2n+5} + \dots \right\} P_n(\cos\theta) \dots (3).$$

2. Let $Y_n(x, y, z)$ denote a spherical harmonic of positive integral degree n, and suppose it is required to evaluate

$$\iint e^{ax+\beta y+\gamma z}\, Y_n(x, y, z)\, dS,$$

where dS is an element of surface of the sphere of radius R, whose centre is the origin, the integral being taken over the whole surface of the sphere. Using the expansion (3), we have

$$e^{ax+\beta y+\gamma z}$$

$$= \Sigma\,(2n+1) \frac{R^n (a^2+\beta^2+\gamma^2)^{\frac{1}{2}n}}{3.5\dots 2n+1} \left\{ 1 + \frac{R^2(a^2+\beta^2+\gamma^2)}{2.2n+3} + \dots \right\} P_n(\cos\theta),$$

where

$$\cos\theta = \frac{ax+\beta y+\gamma z}{R(a^2+\beta^2+\gamma^2)^{\frac{1}{2}}}.$$

If we substitute this expression in the definite integral, since

$$\iint P_m(\cos\theta)\, Y_n(x, y, z)\, dS$$

is zero, unless $m = n$, we have

$$\iint e^{\alpha x+\beta y+\gamma z}\, Y_n(x, y, z)\, dS$$

$$= (2n+1)\,\frac{R^n(\alpha^2+\beta^2+\gamma^2)^{\frac{1}{2}n}}{3.5\ldots 2n+1}\left\{1+\frac{R^2(\alpha^2+\beta^2+\gamma^2)}{2.2n+3}+\ldots\right\}$$

$$\times\iint P_n(\cos\theta)\, Y_n(x, y, z)\, dS.$$

Now $\displaystyle\iint P_n(\cos\theta)\, Y_n(x, y, z)\, dS = \frac{4\pi}{2n+1}\, R^{n+2}\, Y_n\left(\frac{\alpha}{A}, \frac{\beta}{A}, \frac{\gamma}{A}\right)$

$$= \frac{4\pi}{2n+1}\cdot\frac{R^{n+2}}{A^n}\, Y_n(\alpha, \beta, \gamma),$$

A denoting $(\alpha^2+\beta^2+\gamma^2)^{\frac{1}{2}}$; we thus obtain the expression

$$\iint e^{\alpha x+\beta y+\gamma z}\, Y_n(x, y, z)\, dS$$

$$= 4\pi R^{2n+2}\,\frac{2^n n!}{(2n+1)!}\left\{1+\frac{R^2(\alpha^2+\beta^2+\gamma^2)}{2.2n+3}+\frac{R^4(\alpha^2+\beta^2+\gamma^2)^2}{2.4.2n+3.2n+5}+\ldots\right\}$$

$$\times\, Y_n(\alpha, \beta, \gamma) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Now put for α, β, γ the operators $\dfrac{\partial}{\partial x_0}, \dfrac{\partial}{\partial y_0}, \dfrac{\partial}{\partial z_0}$, respectively, and let each side operate upon a function $f(x_0, y_0, z_0)$, where $f(x, y, z)$ is a function which is finite and continuous throughout the volume of the sphere, and where x_0, y_0, z_0 are all put zero after the operations are performed; then, since

$$e^{x.\partial/\partial x_0+y.\partial/\partial y_0+z.\partial/\partial z_0} f(x_0, y_0, z_0)$$

$$= f(x+x_0,\ y+y_0,\ z+z_0) = f(x, y, z),$$

we have the following surface-integral theorem:—

$$\iint Y_n(x, y, z) f(x, y, z)\, dS$$

$$= 4\pi R^{2n+2}\,\frac{2^n n!}{(2n+1)!}\left\{1+\frac{R^2\nabla^2}{2.2n+3}+\frac{R^4\nabla^4}{2.4.2n+3.2n+5}+\ldots\right\}$$

$$Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f(x, y, z) \quad \ldots\ldots\ldots(5),$$

where, on the right-hand side, x, y, z are all put equal to zero after the operations have been performed, and ∇^2 denotes the operator $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$. The only restriction to which the function $f(x, y, z)$ is subject, is that it must be finite and continuous throughout the sphere.

3. I now proceed to consider some particular cases of the theorem (5).

Putting $n=0$, in which case we can put $Y_n=1$, the theorem reduces to that employed by Mr. W. D. Niven,

$$\iint f(x, y, z)\, dS = 4\pi R^2 \left\{1+\frac{R^2\nabla^2}{3!}+\frac{R^4\nabla^4}{5!}+\dots\right\} f(x_0, y_0, z_0) \dots (6).$$

Next suppose that $f(x, y, z)$ is a rational homogeneous function of degree m; in that case the integral vanishes, unless $m-n$ is a positive even number; the theorem then becomes

$$\iint Y_n(x, y, z) f_m(x, y, z)\, dS$$

$$= 4\pi R^{m+n+2}\, 2^n \frac{\left(\frac{m+n}{2}\right)!}{\left(\frac{m-n}{2}\right)!\,(m+n+1)!} \nabla^{m-n} Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f_m(x, y, z) \dots\dots (7),$$

since all the other terms on the right hand vanish.

A particular case of (7) is

$$\iint x^\alpha y^\beta z^\gamma\, Y_n(x, y, z)\, dS$$

$$= 4\pi R^{n+\alpha+\beta+\gamma+2}\, 2^n \frac{\left(\frac{n+\alpha+\beta+\gamma}{2}\right)!}{\left(\frac{\alpha+\beta+\gamma-n}{2}\right)!\,(n+\alpha+\beta+\gamma+1)!}$$

$$\nabla^{\alpha+\beta+\gamma-n}\, Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) x^\alpha y^\beta z^\gamma \dots\dots (8),$$

where $\alpha+\beta+\gamma-n$ is an even integer.

In (7), put $m = n$; we then have

$$\iint Y_n(x, y, z) f_n(x, y, z)\, dS = 4\pi R^{2n+2} \frac{2^n n!}{(2n+1)!} Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f_n(x, y, z) \quad \text{.........................(9).}$$

This last theorem (9) includes, as a particular case, Maxwell's theorem, giving the surface-integral of the product of two surface harmonics of the same degree n. If $h_1, h_2 \ldots h_n$ are the axes of Y_n, we have*

$$Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{(2n)!}{2^n n!\, n!} \frac{\partial^n}{\partial h_1 \partial h_2 \ldots \partial h_n} + \text{a multiple of } \nabla^2,$$

and thus (9) becomes, in the case in which $f_n(x, y, z)$ is a spherical harmonic,

$$\iint Y_n(x, y, z) f_n(x, y, z)\, dS = \frac{4\pi R^{2n+2}}{2n+1} \frac{1}{n!} \frac{\partial^n}{\partial h_1 \partial h_2 \ldots \partial h_n} f_n(x, y, z) \quad \text{...........(10),}$$

which is Maxwell's theorem.† The theorem (9) is more general than Maxwell's, since $f_n(x, y, z)$ is not restricted to being a spherical harmonic, but may be any homogeneous function of degree n.

An important case of (5) is that in which $f(x, y, z)$ is of the form $F(\xi - x, \eta - y, \zeta - z)$, where ξ, η, ζ are the coordinates of a point outside the sphere. In that case, we have

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^s Y_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) F(\xi - x, \eta - y, \zeta - z)$$

$$= (-1)^n \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2}\right)^s Y_n\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right) F(\xi - x, \eta - y, \zeta - z),$$

and, when $x = 0$, $y = 0$, $z = 0$, the expression on the right-hand side becomes

$$(-1)^n \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2}\right)^s Y_n\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right) F(\xi, \eta, \zeta).$$

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* See my paper on "A Theorem in Differentiation," p. 58 of the present volume.

† See *Electricity and Magnetism*, second edition, Vol. I., p. 186.

We thus obtain the following theorem :—

$$\iint Y_n(x, y, z)\, F(\xi - x,\ \eta - y,\ \zeta - z)\, dS$$

$$= 4\pi R^{2n+2}(-1)^n \frac{2^n n!}{(2n+1)!}\left\{1 + \frac{R^2\nabla^2}{2.2n+3} + \frac{R^4\nabla^4}{2.4.2n+3.2n+5} + \dots\right\}$$

$$Y_n\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right) F(\xi, \eta, \zeta) \dots\dots\dots\dots\dots\dots (11),$$

where $$\nabla^2 = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\zeta^2}.$$

If $$\rho^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2,$$

and $$F(\xi - x,\ \eta - y,\ \zeta - z) = \phi(\rho),$$

we obtain the theorem

$$\iint Y_n(x, y, z)\, \phi(\rho)\, dS$$

$$= 4\pi R^{2n+2}(-1)^n \frac{2^n n!}{(2n+1)!}\left\{1 + \frac{R^2\nabla^2}{2.2n+3} + \dots\right\} Y_n\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)\phi(u)$$

$$\dots\dots\dots\dots (12),$$

where $$u^2 = \xi^2 + \eta^2 + \zeta^2.$$

This theorem can be applied to the determination of the potential of a surface distribution on the sphere at an external point, under any law of force; I shall however consider this application in the more general case of a distribution on the surface of an ellipsoid.

If dv is an element of volume of a shell contained between the spheres of radii R and $R + dR$, we have $dv = dS.dR$; hence (12) may be written

$$\iint Y_n(x, y, z)\, \phi(\rho)\, dv$$

$$= 4\pi R^{2n+2} dR\,(-1)^n \frac{2^n n!}{(2n+1)!}\left\{1 + \frac{R^2\nabla^2}{2.2n+3} + \dots\right\} Y_n\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)\phi(u).$$

Multiply both sides by $\psi(R)$, and integrate with respect to R from

$R = 0$ to $R = a$; we then obtain the formula

$$\iiint \psi(R)\, Y_n(x, y, z)\, \phi(\rho)\, dv$$

$$= 4\pi(-1)^n \frac{2^n\, n!}{(2n+1)!}\left\{\int_0^a R^{2n+2}\psi(R)dR + \int_0^a R^{2n+4}\psi(R)dR \frac{\nabla^2}{2.2n+3} + \ldots\right\}$$

$$Y_n\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)\phi(u) \quad \ldots\ldots\ldots\ldots\ldots (13).$$

This volume-integral can be used to obtain the potential of a solid sphere of density $\psi(R)\, Y_n(x, y, z)$ at an external point, under any given law of force.

4. In the fundamental formula (5), put $R = 1$, change x, y, z into $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$ respectively; then the surface integral will be replaced by one taken over the surface of the ellipsoid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

instead of dS, we must write $\frac{p\,dS}{abc}$, where the new dS denotes an element of area of the ellipsoidal surface, and p is the perpendicular from the centre upon the tangent plane containing the element. We thus obtain the formula

$$\iint Y_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) f\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) p\, dS$$

$$= 4\pi abc \frac{2^n\, n!}{(2n+1)!}\left\{1 + \frac{D^2}{2.2n+3} + \ldots\right\} Y_n\left(a\frac{\partial}{\partial x}, b\frac{\partial}{\partial y}, c\frac{\partial}{\partial z}\right)$$

$$f\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right),$$

or, changing $f\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$ into $f(x, y, z)$,

$$\iint Y_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) f(x, y, z)\, p\, dS$$

$$= 4\pi abc \frac{2^n n!}{(2n+1)!}\left\{1 + \frac{D^2}{2.2n+3} + \frac{D^4}{2.4.2n+3.2n+5} + \ldots\right\}$$

$$Y_n\left(a\frac{\partial}{\partial x}, b\frac{\partial}{\partial y}, c\frac{\partial}{\partial z}\right) f(x, y, z) \ldots\ldots (14),$$

where D^2 denotes the operator $a^2\dfrac{\partial^2}{\partial x^2}+b^2\dfrac{\partial^2}{\partial y^2}+c^2\dfrac{\partial^2}{\partial z^2}$; as in (5), x, y, z are put equal to zero when the operations on the right-hand side have been performed.

Corresponding to (11), we obtain, by putting

$$f(x, y, z) = F(\xi-x,\ \eta-y,\ \zeta-z),$$

where ξ, η, ζ are the coordinates of an external point,

$$\iint Y_n\left(\frac{x}{a},\ \frac{y}{b},\ \frac{z}{c}\right) F(\xi-x,\ \eta-y,\ \zeta-z)\, p\, dS$$

$$= 4\pi abc\,(-1)^n \frac{2^n\, n!}{(2n+1)!}\left\{1+\frac{D^2}{2.2n+3}+\dots\right\}$$

$$Y_n\left(a\frac{\partial}{\partial\xi},\ b\frac{\partial}{\partial\eta},\ c\frac{\partial}{\partial\zeta}\right) F(\xi, \eta, \zeta) \quad\text{......(15),}$$

where D^2 now denotes the operator $a^2\dfrac{\partial^2}{\partial\xi^2}+b^2\dfrac{\partial^2}{\partial\eta^2}+c^2\dfrac{\partial^2}{\partial\zeta^2}$.

Corresponding to (12), we obtain

$$\iint Y_n\left(\frac{x}{a},\ \frac{y}{b},\ \frac{z}{c}\right) \phi(\rho)\, p\, dS$$

$$= 4\pi abc\,(-1)^n \frac{2^n n!}{(2n+1)!}\left\{1+\frac{D^2}{2.2n+3}+\dots\right\}$$

$$Y_n\left(a\frac{\partial}{\partial\xi},\ b\frac{\partial}{\partial\eta},\ c\frac{\partial}{\partial\zeta}\right) \phi\left(\sqrt{\xi^2+\eta^2+\zeta^2}\right) \text{...(16),}$$

where $\qquad \rho^2 = (\xi-x)^2+(\eta-y)^2+(\zeta-z)^2.$

In (16), change a, b, c into $\epsilon a, \epsilon b, \epsilon c$; then

$$\delta p\,.\,dS = \frac{\delta\epsilon}{\epsilon}\,.\,p\,dS = dv$$

is the element of volume of a shell bounded by the two ellipsoids whose semi-axes are ϵa, ϵb, ϵc and $(\epsilon+d\epsilon)\,a$, $(\epsilon+d\epsilon)\,b$, $(\epsilon+d\epsilon)\,c$ respectively. Multiplying both sides of the equation by $\epsilon^{n-1}\psi(\epsilon)\,d\epsilon$,

and integrating from $\epsilon = 0$ to $\epsilon = 1$, we obtain the formula

$$\iiint Y_n\left(\frac{x}{a},\ \frac{y}{b},\ \frac{z}{c}\right)\psi(\epsilon)\,\phi(\rho)\,dv$$

$$= 4\pi abc\,(-1)^n\frac{2^n n!}{(2n+1)!}\left\{\int_0^1 \epsilon^{2n+2}\psi(\epsilon)\,d\epsilon+\int_0^1 \epsilon^{2n+4}\,\psi(\epsilon)\,d\epsilon\,\frac{D^2}{2.2n+3}+\dots\right\}$$

$$Y_n\left(a\frac{\partial}{\partial\xi},\ b\frac{\partial}{\partial\eta},\ c\frac{\partial}{\partial\zeta}\right)\phi\left(\sqrt{\xi^2+\eta^2+\zeta^2}\right)\dots(17),$$

where the volume-integral is taken throughout the volume of the ellipsoid, and ϵ denotes $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{2}}$.

5. The first application that I shall make of the formulae of the last section is to express an external ellipsoidal harmonic in a series of spherical harmonics; I use throughout the notation in Mr. W. D. Niven's memoir* on ellipsoidal harmonics, in which memoir the expression is found by other methods.

At the surface of the ellipsoid, the ellipsoidal harmonic

$$G_n(x, y, z) \text{ or } \left\{\begin{matrix} & x & yz & \\ 1 & y & zx & xyz \\ & z & xy & \end{matrix}\right\}\Pi\left(\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}-1\right)$$

is equal to

$$\Pi(-\theta)\left\{\begin{matrix} & a^{-1}x & b^{-1}c^{-1}yz & \\ 1 & b^{-1}y & c^{-1}a^{-1}zx & a^{-1}b^{-1}c^{-1}xyz \\ & c^{-1}z & a^{-1}b^{-1}xy & \end{matrix}\right\}\times\left\{\begin{matrix} & a & bc & \\ 1 & b & ca & abc \\ & c & ab & \end{matrix}\right\}$$

$$\times\,\Pi\left(\frac{x^2}{a^2.\,a^2+\theta}+\frac{y^2}{b^2.\,b^2+\theta}+\frac{z^2}{c^2.\,c^2+\theta}\right),$$

or
$$\left\{\begin{matrix} & a & bc & \\ 1 & b & ca & abc \\ & c & ab & \end{matrix}\right\}\Pi(-\theta)\,H_n\left(\frac{x}{a},\ \frac{y}{b},\ \frac{z}{c}\right),$$

where $H_n(x, y, z)$ is a spherical harmonic.

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* *Philosophical Transactions*, 1891.

In (16), put $Y_n = H_n$, $\phi(\rho) = \dfrac{1}{\rho}$; we then have

$$\iint H_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)\frac{1}{\rho}\,p\,dS$$

$$= 4\pi abc\,(-1)^n\frac{2^n\,n!}{(2n+1)!}\left\{1+\frac{D^2}{2.2n+3}+\dots\right\}$$

$$H_n\left(a\frac{\partial}{\partial\xi}, b\frac{\partial}{\partial\eta}, c\frac{\partial}{\partial\zeta}\right)\frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}};$$

now
$$H_n\left(a\frac{\partial}{\partial\xi}, b\frac{\partial}{\partial\eta}, c\frac{\partial}{\partial\zeta}\right)\frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}}$$

$$=\left\{\begin{matrix} & a & bc & \\ 1 & b & ca & abc \\ & c & ab & \end{matrix}\right\}\Pi(-\theta)\,H_n\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)\frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}},$$

and thus we have

$$\iint \frac{1}{\rho}\,G_n(x, y, z)\,p\,dS$$

$$= 4\pi abc\kappa^2\{\Pi(\theta)\}^2\frac{2^n\,n!\,(-1)^n}{(2n+1)!}\left\{1+\frac{D^2}{2.2n+3}+\dots\right\}$$

$$\times H_n\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)\frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}},$$

where κ denotes the bracket containing a, b, c.

Now, if $\mathfrak{G}_n(\xi, \eta, \zeta) = G_n(\xi, \eta, \zeta)\,I_n(\xi, \eta, \zeta)$ denotes an external harmonic, where I_n is an integral of the form

$$\int_\epsilon^\infty \frac{d\lambda}{(\theta_1-\lambda)^2(\theta_2-\lambda)^2\dots(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{1}{2}}},$$

ϵ being the parameter of the confocal ellipsoid through the point ξ, η, ζ, the surface density σ of a distribution on the ellipsoid which will produce an external potential $\mathfrak{G}_n(\xi, \eta, \zeta)$, is given by

$$4\pi\sigma = -\frac{\partial\mathfrak{G}_n}{\partial\nu}+I_n\frac{\partial G_n}{\partial\nu},$$

where $\partial\nu$ is an element of normal, and I_n has its value at the surface of the ellipsoid; we thus obtain

$$4\pi\sigma = G_n(x, y, z)\frac{\partial\epsilon}{\partial\nu}\cdot\frac{1}{\{\Pi(\theta)\}^2}\cdot\frac{1}{abc\kappa^2},$$

and it is easily shown that $\dfrac{\partial \epsilon}{\partial \nu} = 2p$;

hence $$\sigma = \frac{pG_n(x, y, z)}{2\pi \{\Pi(\theta)\}^2} \cdot \frac{1}{abc\kappa^2}.$$

We obtain therefore the formula

$$\mathfrak{G}_n(\xi, \eta, \zeta) = (-1)^n \frac{2^{n+1} n!}{(2n+1)!} H_n\left(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \zeta}\right)$$

$$\times \left\{1 + \frac{D^2}{2.2n+3} + \frac{D^4}{2.4.2n+3.2n+5} + \dots\right\} \frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}} \quad \dots\dots\dots\dots\dots\dots(18),$$

which is Mr. Niven's expression for an external ellipsoidal harmonic in a series of spherical harmonics. It will be observed that in the above proof the distance of the point ξ, η, ζ from the origin is not necessarily greater than the greatest semi-axis of the ellipsoid, for, since (12) holds for all points external to the sphere, it follows that (16) and consequently (18) holds for all points external to the ellipsoid.

6 When it is required to find the potentials of ellipsoidal shells or of solid ellipsoids of variable density, at an external point, it is in ordinary cases better not to use the ellipsoidal harmonics, but to use the theorems (16) and (17). The formula (16) gives an expression for the potential of a surface distribution of which the density is $pY_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$, when the law of force is $-\phi'(\rho)$; in order to find the potential of a surface distribution of density $pF(x, y, z)$, it is necessary to express $F(x, y, z)$ as the sum of a number of functions of the form $Y_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$.

The formula (17) gives the potential, at an external point, of a solid ellipsoid whose density is

$$Y_n\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) \psi\left(\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}\right),$$

and thus, as in the case of a shell, the potential of an ellipsoid whose density is

$$F(x, y, z)\,\chi\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$$

can be found, the law of attraction being $-\phi'(\rho)$. I shall obtain the formulae for the potential in one or two simple cases, as an example of the general method.

(a) To find the potential of a homeoid whose density is μxyz, the law of force being that of the inverse square: in this case xyz is a harmonic of degree $n = 3$; we thus obtain, from (16),

$$V = -\frac{4\pi a^3 b^3 c^3 \mu}{105}\left\{1 + \frac{D^2}{2.9} + \frac{D^4}{2.4.9.11} + \dots\right\}\frac{\partial^3}{\partial\xi\,\partial\eta\,\partial\zeta}\cdot\frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}},$$

where
$$D^2 = a^2\frac{\partial^2}{\partial\xi^2} + b^2\frac{\partial^2}{\partial\eta^2} + c^2\frac{\partial^2}{\partial\zeta^2}.$$

(b) To find the potential of a solid gravitating ellipsoid whose density is
$$\mu z^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^m,$$

where m is any positive quantity: in this case we write μz^2 in the form
$$\frac{\mu c^2}{3}\left\{\left(\frac{2z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) + \epsilon^2\right\},$$

the quantity $\frac{2z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2}$ being of the form
$$Y_2\left(\frac{x}{a},\ \frac{y}{b},\ \frac{z}{c}\right).$$

The required potential is the sum of the potentials of the ellipsoids whose densities are $\frac{\mu c^2}{3}\left(\frac{2z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\epsilon^{2m}$, and $\frac{\mu c^2}{3}\epsilon^{2m+2}$; we thus obtain, from (17), for the potential required,

$$V = \tfrac{4}{3}\pi\mu abc^3\,\Sigma\,\frac{1}{(2t+3)(2t+5)(2t+2m+5)(2t+1)!}$$
$$D^{2t}\left(2c^2\frac{\partial^2}{\partial\zeta^2} - a^2\frac{\partial^2}{\partial\xi^2} - b^2\frac{\partial^2}{\partial\eta^2}\right)\frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}}$$
$$+\tfrac{4}{3}\pi\mu abc^3\,\Sigma\,\frac{1}{(2t+2m+5)(2t+1)!}\,D^{2t}\,\frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}}.$$

(c) To find the potential of a solid gravitating ellipsoid whose density is
$$\mu\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)^{m-1},$$

where m is a positive quantity: we have

$$\int_0^1 \epsilon^{2t+2}(1-\epsilon^2)^{m-1}\,d\epsilon = \frac{\Gamma(t+\frac{3}{2})\,\Gamma(m)}{2\Gamma(m+t+\frac{3}{2})},$$

so that the required potential is

$$V = 2\pi abc \sum_{t=0}^{t=\infty} \frac{\Gamma(t+\frac{3}{2})\,\Gamma(m)}{\Gamma(m+t+\frac{3}{2})(2t+1)!} D^{2t} \frac{1}{\sqrt{\xi^2+\eta^2+\zeta^2}}.$$

7. I shall now proceed to modify the formula (17), so that it may be adapted to the case in which the ellipsoid becomes an elliptic disc; we put $c = 0$, and in this case we must suppose that $Y_n(x, y, z)$ does not contain z, or that $Y_n(x, y)$ is one of the harmonics

$$(x+\iota y)^n+(x-\iota y)^n, \quad \iota\left\{(x+\iota y)^n-(x-\iota y)^n\right\}.$$

The mass of a prismatic section of the ellipsoid of which the base is the element $dxdy$, is

$$2c\,dx\,dy \int_\alpha^1 \frac{\psi(\epsilon)\,\epsilon}{\sqrt{\epsilon^2-\alpha^2}}\,d\epsilon,$$

where

$$\alpha^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}.$$

If we put $\chi(\alpha)$ for the value of the definite integral, we find, on dividing both sides of the equation (17) by $2c$,

$$\iint \left(\frac{x}{a} \pm \iota\frac{y}{b}\right)^n \chi(\alpha)\,\phi(\rho)\,dx\,dy$$

$$= 2\pi ab(-1)^n \frac{2^n n!}{(2n+1)!}\left\{\int_0^1 \epsilon^{2n+2}\psi(\epsilon)\,d\epsilon + \int_0^1 \epsilon^{2n+4}\psi(\epsilon)\,d\epsilon\,\frac{D^2}{2.2n+3}+\dots\right\}$$

$$\left(a\frac{\partial}{\partial\xi} \pm \iota b\frac{\partial}{\partial\eta}\right)^n \phi\left(\sqrt{\xi^2+\eta^2+\zeta^2}\right) \dots (19),$$

where

$$D^2 = a^2\frac{\partial^2}{\partial\xi^2}+b^2\frac{\partial^2}{\partial\eta^2}.$$

This formula gives the potential of an elliptic disc of density $\left(\frac{x}{a} \pm \iota\frac{y}{b}\right)^n \chi\left(\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\right)$ at an external point, the law of force being $-\phi'(\rho)$.

As an example of the use of (19), suppose

$$\psi(\epsilon) = (1-\epsilon^2)^{m-\frac{1}{2}};$$

then
$$\chi(a) = \int_a^1 \frac{(1-\epsilon^2)^{m-\frac{1}{2}}}{\sqrt{\epsilon^2-a^2}}\,\epsilon\,d\epsilon.$$

Changing the variable in the integration to v, where

$$(1-\epsilon^2) = v\,(1-a^2),$$

we have
$$\chi(a) = \tfrac{1}{2}(1-a^2)^m \int_0^1 v^{m-\frac{1}{2}}(1-v)^{-\frac{1}{2}}\,dv;$$

thus
$$\chi(a) = \tfrac{1}{2}(1-a^2)^m \frac{\Gamma(m+\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(m+1)} = \frac{\pi}{2}(1-a^2)^m \frac{(2m)!}{2^{2m}(m!)^2}.$$

Putting $n=0$, in (19) we find, for the potential of an elliptic disc of uniform thickness, and of density $\mu\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^m$, the value

$$V = 2^{2m+2}\frac{(m!)^2}{(2m)!}\mu ab \sum_{t=0}^{t=\infty}\int_0^1 \epsilon^{2t+2}(1-\epsilon^2)^{m-\frac{1}{2}}\,d\epsilon\,\frac{D^{2t}}{(2t+1)!}\,\phi\left(\sqrt{\xi^2+\eta^2+\zeta^2}\right)$$

$$= m!\,\mu\pi ab \sum_{t=0}^{t=\infty}\frac{1}{(t+m+1)!\,t!\,2^{2t}}\left(a^2\frac{\partial^2}{\partial\xi^2}+b^2\frac{\partial^2}{\partial\eta^2}\right)^t \phi\left(\sqrt{\xi^2+\eta^2+\zeta^2}\right).$$

As another example, suppose it is required to find the potential of a disc of density $\mu\,xy\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^m$; in this case we put $n=2$, since xy is a harmonic of the second degree. The values of $\psi(\epsilon)$ and $\chi(a)$ are the same as before; we have therefore

$$V = 2^{2m+2}\frac{(m!)^2}{(2m)!}\mu a^3b^3\frac{2^2.2!}{5!}\sum_{t=0}^{t=\infty}\int_0^1 \epsilon^{2t+4}(1-\epsilon^2)^{m-\frac{1}{2}}\,d\epsilon$$

$$\frac{15D^{2t}}{(2t+1)!\,(2t+3)(2t+5)}\,\frac{\partial^2}{\partial\xi\partial\eta}\,\phi\left(\sqrt{\xi^2+\eta^2+\zeta^2}\right)$$

$$= \tfrac{1}{4}m!\,\mu\pi a^3b^3\sum_{t=0}^{t=\infty}\frac{1}{t!\,(t+m+2)!\,(2t+5)\,2^{2t}}$$

$$\times\left(a^2\frac{\partial^2}{\partial\xi^2}+b^2\frac{\partial^2}{\partial\eta^2}\right)^t\frac{\partial^2}{\partial\xi\partial\eta}\,\phi\left(\sqrt{\xi^2+\eta+\zeta^2}\right).$$

8. A line-integral round the circumference of a circle, analogous to the surface integral in (4), may be found.

We have $$e^{\iota\rho\cos\phi} = J_0(\rho) + 2\sum_{n=1} \iota^n J_n(\rho)\cos n\phi,$$

where $$J_n(\rho) = \frac{\rho^n}{2^n n!}\left\{1 - \frac{\rho^2}{2.2n+2} + \frac{\rho^4}{2.4.2n+2.2n+4} - \dots\right\};$$

thence we have

$$e^{\alpha x+\beta y} = \left\{1 + \frac{R^2(\alpha^2+\beta^2)}{2^2} + \frac{R^4(\alpha^2+\beta^2)^2}{2^2.4^2} + \dots\right\}$$

$$+\sum \frac{2R^n(\alpha^2+\beta^2)^{\frac{1}{2}n}}{2^n n!}\left\{1 + \frac{R^2(\alpha^2+\beta^2)}{2.2n+2} + \dots\right\}\cos n\phi,$$

where $x^2+y^2 = R^2$, and $\cos\phi = \dfrac{\alpha x+\beta y}{R\sqrt{\alpha^2+\beta^2}} = \cos(\theta-\beta)$,

where $x = R\cos\theta$, $y = R\sin\theta$, $\cos\beta = \dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}}$, $\sin\beta = \dfrac{\beta}{\sqrt{\alpha^2+\beta^2}}$.

The value of the integral $\int e^{\alpha x+\beta y}(x\pm\iota y)^n\,ds$, taken round the circumference of a circle of radius R, whose centre is at the origin, is easily seen to be

$$2\pi R^{2n+1}\frac{1}{2^n n!}\left\{1 + \frac{R^2(\alpha^2+\beta^2)}{2.2n+2} + \frac{R^4(\alpha^2+\beta^2)^2}{2.4.2n+2.2n+4} + \dots\right\}(\alpha\pm\iota\beta)^n.$$

As before, we obtain from this result the theorem

$$\int (x\pm\iota y)^n f(x, y)\,ds$$

$$= 2\pi R^{2n+1}\frac{1}{2^n n!}\left\{1 + \frac{R^2\nabla^2}{2.2n+2} + \frac{R^4\nabla^4}{2.4.2n+2.2n+4} + \dots\right\}$$

$$\left(\frac{\partial}{\partial x} \pm \iota\frac{\partial}{\partial y}\right)^n f(x, y) \quad\dots\dots(20),$$

where, on the right-hand side, x and y are put equal to zero after the operations are performed; $f(x, y)$ must be finite and continuous within the circle, and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

If $$f(x, y) = F(\xi - x, \eta - y),$$

where ξ, η, ζ are the coordinates of an external point, (20) becomes

$$\int (x \pm \iota y)^n F(\xi - x, \eta - y, \zeta)\, ds$$

$$= 2\pi R^{2n+1} \frac{(-1)^n}{2^n n!} \left\{ 1 + \frac{R^2 \nabla^2}{2.2n+2} + \dots \right\} \left(\frac{\partial}{\partial \xi} \pm \iota \frac{\partial}{\partial \eta} \right)^n F(\xi, \eta, \zeta) \dots (21),$$

where $$\nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}.$$

The theorem for the ellipse which can be derived from (20), is

$$\int \left(\frac{x}{a} \pm \iota \frac{y}{b} \right)^n f(x, y)\, p\, ds$$

$$= 2\pi ab \frac{1}{2^n n!} \left\{ 1 + \frac{D^2}{2.2n+2} + \frac{D^4}{2.4.2n+2.2n+4} + \dots \right\}$$

$$\times \left(a \frac{\partial}{\partial x} \pm \iota b \frac{\partial}{\partial y} \right) f(x, y) \dots (22),$$

where $$D^2 = a^2 \frac{\partial^2}{\partial x^2} + b^2 \frac{\partial^2}{\partial y^2},$$

and, as before, the value of the expression on the right-hand side is taken at the origin. In the special case

$$f(x, y) = F(\xi - x, \eta - y, \zeta);$$

this becomes $$\int \left(\frac{x}{a} \pm \iota \frac{y}{b} \right)^n F(\xi - x, \eta - y, \zeta)\, p\, ds$$

$$= 2\pi ab \frac{(-1)^n}{2^n n!} \left\{ 1 + \frac{D^2}{2.2n+2} + \dots \right\} \left(a \frac{\partial}{\partial \xi} \pm \iota b \frac{\partial}{\partial \eta} \right)^n F(\xi, \eta, \zeta) \dots (23),$$

where $$D^2 = a^2 \frac{\partial^2}{\partial \xi^2} + b^2 \frac{\partial^2}{\partial \eta^2}.$$

We might proceed to obtain, from these last results, integrals taken over the area of the ellipse: such integrals we have, however, obtained as a special case of the ellipsoidal volume-integrals; it is therefore unnecessary to proceed further in this direction.

On the application of the Sylvester-Clifford Graphs to Ordinary Binary Quantics. (*Second Part.*) *By* A. B. KEMPE, M.A., F.R.S. Received and read January 12th, 1893.

This paper is a continuation of that "On the Application of Clifford's Graphs to Ordinary Binary Quantics," printed in the *Proceedings*, Vol. XVII., pp. 107–121, and the divisions and sections are accordingly numbered continuously with those of the preceding part. An alteration has, however, been made in the title. This has been done because, though the subject is regarded from the point of view of Professor Clifford, and therefore, following the precedent set by the late Mr. Spottiswoode in his paper "On Clifford's Graphs" (*Proceedings*, Vol. X., p. 204), his name alone was in the first part attached to the "graphs," it has been thought, on reflection, that by such exclusive association, an impression might be created which would operate unjustly towards the unquestionable originality of the paper by Professor Sylvester "On an application of the new Atomic Theory to the Graphical Representation of the Invariants and Covariants of Binary Quantics" (*American Journal of Mathematics*, Vol. I., p. 64). With regard to the "singularly beautiful series of conceptions" relative to these graphical representations, the late Professor H. J. S. Smith, in his introduction to Clifford's "Mathematical Papers" (Macmillan & Co., 1882), remarks that "it is difficult to say how much belongs to Clifford and how much to Sylvester, the more so because each of them was ready to attribute to the other the largest share." Under these circumstances the proper course would seem to be that, here adopted, of designating the graphs by their joint names.

In the former part it was shown that on multiplying any invariant or covariant of a binary quantic by a proper number of polar elements, it can be expressed as the sum of one or more "pure compound forms," each of which is itself an invariant or covariant of such quantic multiplied by the same polar elements; that to the algebraical expression of a pure compound form there corresponds directly an equivalent "graph"; that these graphs admit of being multiplied and added together, and the syzygetical relations between invariants and covariants of being expressed by purely graphical formulæ. In this second part an alternative algebraical method of representing compound forms is given, which, though directly

derivable from the preceding method, and suggested by the graphs, is of a totally distinct character, and is independent of the use of polar elements. It is also shown that seminvariants can be directly represented by compound forms as well as invariants and covariants, a result due to the fact that the *seminvariants* of any quantic $(a_0, a_1, a_2, a_3, \ldots a_n \between x, y)^n$ are *invariants* of two or more of the series of quantics

$$a_0\,;\ (a_0, a_1 \between x, y)^1\,;\ (a_0, a_1, a_2 \between x, y)^2\,;\ \ldots\ (a_0, a_1, a_2, a_3, \ldots a_n \between x, y)^n.$$

The expression of pure compound forms as the sum of products of simpler pure compound forms is also considered, and a theorem given with regard to it. Some other matters are also briefly dealt with which need not be specifically referred to here.

V. *Order, Degree, and Weight of Pure Compound Forms.*

57. It will be as well to point out that a pure compound form (Secs. 17, 24) which contains

j simple forms of mark a and valence i,
w bonds connecting these forms to each other,
g simple forms of mark x, and valence 1,
g bonds connecting these simple forms of mark x to those of mark a,

represents a covariant of

order g in the variables,
degree j in the coefficients,
weight w in the coefficients,

of the quantic $(a_0, a_1, a_2, \ldots a_i \between x, y)^i$ of order i in the variables.

58. Since every bond connects two simple forms, and there are j simple forms of valence i, and g simple forms of valence 1, the total number of bonds will be

$$\frac{ij+g}{2}.$$

But, since w bonds connect the simple forms of mark a to each other, and g bonds connect the simple forms of mark a to those of mark x, and there are no bonds connecting the simple forms of mark x to each other, the total number of bonds is also

$$w+g.$$

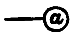
Thus we have $ij+g = 2w+2g$, and therefore $g = ij-2w$.

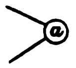
59. If we do not distinguish between the variables and the coefficients, since the total number of simple forms in the compound form is $j+g$, and the total number of bonds is $w+g$, we may speak of the pure compound form as of degree $j+g$ and of weight $w+g$, and in future when the *degree* and *weight* of a pure compound form are spoken of, it is to be understood that reference is made to the total number of simple forms and bonds respectively which compose it. When the simple forms are all of the same valence i, we may speak of the pure compound form as of *order* i.

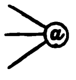
VI. *Simple Forms of Like Marks but Different Valence.*

60. In Sec. 6 it is assumed that simple forms which differ in valence must also differ as to their marks. No conclusions are, however, deduced from this assumption, beyond the statements in Secs. 19 and 22, that where simple forms differ in valence it is unnecessary to indicate specifically that they have different marks. The assumption is erroneous, for we may obviously have the infinite series of forms

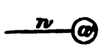

$$= (\ )_a = a_0,$$


$$= (u)_a = a_0 U + a_1 U',$$


$$= (uv)_a = a_0 UV + a_1 (U'V + UV') + a_2 U'V',$$


$$= (uvw)_a = a_0 UVW + a_1 (U'VW + UV'W + UVW') + a_2 (UV'W' + U'VW' + U'V'W) + a_3 U'V'W',$$

... ...


$$= (uvw \dots \text{to } n \text{ letters})_a = \Sigma_{r=n}^{r=0} a_r \left[uvw \dots \text{to } n \text{ letters}\right]^r$$

(see Secs. 1, 2, 3),

where observe that

$$U'(uvw \dots)_a = U'U(vw \dots)_a,$$

and therefore $\quad U'(u)_a = U'U(\ )_a;$

whence, since $\quad U'(u)_a = U'Ua_0,$

we have $\quad (\ )_a = a_0,$

as given.

61. Consider now the two special linear forms

 $= (u)_\epsilon = U'$, so that $\epsilon_0 = 0$ and $\epsilon_1 = 1$;

 $= (u)_\eta = U$, so that $\eta_0 = 1$ and $\eta_1 = 0$.

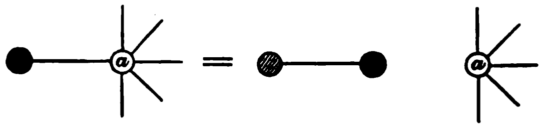
We have  $= (u)_\eta\,(u)_\epsilon = UU'$,

a pure compound form of which the coefficient (Sec. 17) is 1. Also, since

$$U'(uvw\ldots)_a = U'U\,(vw\ldots)_a,$$

we have $$(u)_\epsilon\,(uvw\ldots)_a = (u)_\epsilon\,(u)_\eta\,(vw\ldots)_a;$$

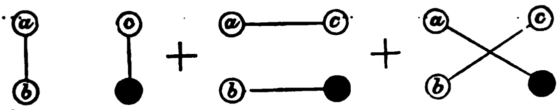
and therefore



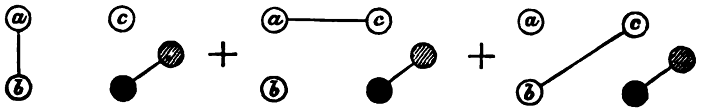
Hence, by multiplying any compound form by $(u)_\epsilon\,(u)_\eta$, we can, without altering the value of its coefficient, increase the valence of any of its component simple forms by 1, and the weight also by 1; and we can repeat the process at will.

62. On the other hand, in cases where we have forms such as that on the left-hand side of the identity of the last section, and we are concerned only with the coefficients of the pure compound forms under consideration, the linear form  and its bond may be omitted without affecting such coefficients.

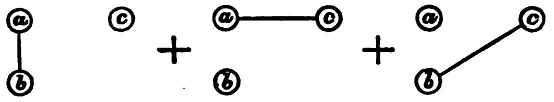
63. Again, by the formula of Sec. 30, we have

 $= 0$;

where, as stated in that section, there may be any number of bonds and nuclei besides those represented. It follows from the formula of Sec. 61 that this may be expressed as

 $= 0$;

or, removing the common factor $(u)_c\,(u)_v$, as



$$= 0,$$

a formula giving syzygetical relations between compound forms in which the corresponding simple forms, though of like marks, may differ in valence in different terms; *e.g.*, in the formula, a is of lower valence in the third term than it is in the first two. Formulæ such as the first and last of this section may be spoken of as *homogeneous* and *non-homogeneous* respectively. Great care must be exercised in the use of non-homogeneous formulæ, for, owing to the fact that there is no such correspondence of the points of egress of the bonds in the terms as exists in the case of the homogeneous formulæ, difficulties arise as to the determination of the signs of the terms (Sec. 29). In cases, however, where we are only concerned to show that certain compound forms may be expressed as the sums of others of certain special descriptions, and the signs are not material, the formulæ may be used freely without special care. The signs, when necessary, can be accurately determined by expressing the formulæ algebraically (see note to Sec. 69).

VII. *Algebraical Representation of Forms. Second Method.*

64. I proceed to indicate a second algebraical method of representing forms, suggested to me by the graphical method, and directly derivable from the first algebraical method. I propose to call these two algebraical methods the "polar" and "difference" methods respectively. Some reference is made to the latter method by Mr. S. Roberts in his paper "Concerning Semi-invariants," *Proceedings*, Vol. XXI., p. 220, where he states that he had been informed by Mr. Hammond that it had been given by me. I had not, however, published anything on the subject, though I had made verbal communications with respect to it to Mr. Hammond and others. Mr. Roberts states the method in a more restricted form than that in which it has always presented itself to me, and makes no reference to its relation to the graphical mode of representation. I therefore venture to set it forth here, and to indicate some applications of it.

65. The method deals with *pure* compound forms (Sec. 17), and in the first instance with those only which are *primary* (Sec. 15). The nuclei of the corresponding graphs will accordingly be supposed to have each a different mark, and every bond will proceed from one nucleus to another, there being no *free* bonds (Sec. 24).

66. Let
$$a = a_1 \frac{d}{da_0} + a_2 \frac{d}{da_1} + a_3 \frac{d}{da_2} + \dots \textit{ ad inf.},$$
$$b = b_1 \frac{d}{db_0} + b_2 \frac{d}{db_1} + b_3 \frac{d}{db_2} + \dots \textit{ ad inf.},$$
and similarly in the case of other letters c, d, e, &c. Then we have
$$a_r = a^r a_0, \quad a_r b_s = a^r b^s a_0 b_0.$$
We may therefore write the formula of Sec. 3,
$$(uvw\dots)_a$$
$$= \left\{ \left[uvw\dots\right]^0 + a\left[uvw\dots\right]^1 + a^2\left[uvw\dots\right]^2 + \dots + a^n\left[uvw\dots\right]^n \right\} a_0{}^*$$
$$= \left\{U + U'a\right\}\left\{V + V'a\right\}\left\{W + W'a\right\} \dots a_0;$$
and thus a simple form may be expressed as the product of a number of terms such as $\left\{U + U'a\right\}$ operating upon a_0.

67. A pure primary compound form, being merely the product of a number of simple forms, may therefore be written
$$\left\{U + U'a\right\}\left\{V + V'a\right\} \dots \left\{U + U'f\right\} \dots \left\{V + V'g\right\} \dots a_0 b_0 c_0 d_0 e_0 f_0 g_0 \dots,$$
where, since the form is pure, to every factor $\left\{U + U'a\right\}$ there corresponds a factor $\left\{U + U'f\right\}$, and one only; and, since it is primary, a and f are different letters. But
$$\left\{U + U'a\right\}\left\{U + U'f\right\} = \left\{f - a\right\} UU'.$$
Hence a pure primary form may be written as
$$\left\{a - b\right\}^\lambda \left\{b - c\right\}^\mu \left\{a - c\right\}^\nu \dots a_0 b_0 c_0 \dots UU'VV'WW' \dots,$$
the exponent λ being the same as the number of the bonds joining the two simple forms whose marks are a and c respectively, and similarly in the case of the exponents of the other factors. This mode of representing a primary compound form is that which I term "the difference method."

68. The expression of a form by the difference method consists then of three sets of factors, viz.,

(1) The difference, or operating, factors, such as $\left\{a - b\right\}^\lambda$.

(2) The leading, or operand, factors, such as a_0, b_0.

(3) The polar elements, or bond coordinates (Sec. 6), such as U, U'.

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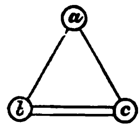
* Compare the foot-note to Sec. 10, where the operator a is denoted by D_a. If we *there put* $A = uvw\dots$, and $\left\{a_0\left[B\right]^0 + a_1\left[B\right]^1 + a_2\left[B\right]^2 + \dots\right\} \equiv (B)_a = (\dots)_a \equiv a_0$ *(Sec. 60), we obtain the formula of the text.*

69. It will be interesting, for the purpose of comparison, to give the equivalent graphical, polar, difference, and ordinary algebraical representations of a particular form. We have



$$= (uv)_a\,(uwx)_b\,(vwx)_c$$
$$= (a-b)(b-c)^2\,(a-c)\,a_0b_0c_0\,.\,UU'VV'WW'XX'$$
$$= \{a_2b_2c_0-2a_2b_1c_1+a_2b_0c_2-a_1b_2c_0+a_1b_2c_1+a_1b_1c_2-a_1b_0c_2$$
$$+a_0b_2c_1-2a_0b_2c_2+a_0b_1c_2\}\;UU'VV'WW'XX'.$$

It will be observed that in the polar method the factors represent the nuclei, or simple forms, of the graphical representation,* while in the difference method the factors of the essential or distinctive part—the operator—correspond to the bonds.

70. It is obvious that, though the polar elements are necessary factors in expressing the equivalence of the polar representation and the differential one, if we confine ourselves to the difference method we need not trouble about the polar elements, and may regard our graphs as representing the operator and operand terms only; *e.g.*, we may regard the graph of the last section as equal to $(a-b)(b-c)^2\,(a-c)\,a_0b_0c_0$. Occasionally it will be convenient, instead of omitting the polar elements, to use the symbol Π to denote the factor which is the product of the polar elements, however many of such polar elements there may be. Thus, if C_0 be the coefficient of any pure compound form C, we shall have $C = C_0\Pi$.

71. In some cases it will be unnecessary to express either the polar elements or the operands a_0, b_0, c_0, ..., and we may regard the graphs as representing merely the product of a number of differences of operators. Here the operators may be regarded just as if they were ordinary quantities, and the discussion of pure primary forms reduces itself to the discussion of expressions which are the product of differences of quantities—say, are difference terms—and the relations of such expressions to each other. It may be here pointed out that the fundamental homogeneous syzygetical identity of Sec. 30

* It is obvious that there is nothing to determine whether the bond joining a nucleus of mark a to one of mark b corresponds to a factor $a-b$ or a factor $b-a$. It has been suggested by Mr. Hammond that the correspondence may be made determinate by the addition of an arrowhead to each bond, with the convention that where the barb points from a to b, the bond shall be understood to correspond to the factor $a-b$, and *vice versâ*. The adoption of this plan would remove the difficulty *as to signs, referred to* in Secs. 29 and 63.

will, in the difference method, be expressed by

$$(a-b)(c-d)+(a-c)(d-b)+(a-d)(b-c) \equiv 0,$$

while the non-homogeneous one of Sec. 63 will be expressed by

$$(a-b)+(b-c)+(c-a) \equiv 0.$$

72. Observe that, since

$$(\epsilon-a)\, a_0\epsilon_0 = (\epsilon_1 - a\epsilon_0)\, a_0 = \epsilon_1 a_0 = \epsilon \,.\, a_0\epsilon_0 \quad \text{(Sec. 61)},$$

we have $\qquad \epsilon - a \equiv \epsilon.$

Also, since $\qquad (a-\eta)\, a_0\eta_0 = (a\eta_0 - \eta_1)\, a_0 = a \,.\, a_0\eta_0,$

we have $\qquad a - \eta \equiv a.$

73. The difference method is not directly applicable to the case of a degraded primary form (Sec. 16); but it can be made applicable by taking the corresponding primary form from which the degraded form is derivable (*i.e.*, the primary form obtained by replacing like marks on the degraded form by unlike), and appending an indication that the unlike marks are, after the operator has operated upon the operand $a_0 b_0 c_0 \ldots$, to be replaced by like marks. Thus the form $(uv)_a\,(uwx)_b\,(vwx)_b$ may be represented in the difference method by

$$(a-b)(b-c)^2(a-c)\, a_0 b_0 c_0 \Pi, \quad (b=c);$$

the indication $b = c$ being construed to mean that after the operator $(a-b)(b-c)^2(a-c)$ has operated upon $a_0 b_0 c_0$, we are to put

$$b_0 = c_0, \quad b_1 = c_1, \quad b_2 = c_2, \quad b_3 = c_3.$$

If this be done, we get

$$2\left\{a_2 b_2 b_0 - a_2 b_1^2 + a_1 b_1 b_2 - a_1 b_3 b_0 + a_0 b_3 b_1 - a_0 b_2^2\right\} \Pi.$$

74. For some purposes, *e.g.*, where we are considering the syzygetical relations between compound forms, it is sufficient to regard the direction $b = c$ as meaning that in the operator T to which it applies, or in any one or more of the terms in any expression which is equal to T, we may interchange c and b without affecting the value of T.

75. In accordance with the notation of Sec. 66, we have

$$x = x_1 \frac{d}{dx_0} + x_2 \frac{d}{dx_1} + \ldots,$$

$$y = y_1 \frac{d}{dy_0} + y_2 \frac{d}{dy_1} + \ldots.$$

Here the x, y, are not, of course, the x, y of the quantic which in the

ordinary notation would be written

$$(a_0, a_1, a_2, \dots a_n \between x, y)^n,$$

these latter being respectively the x_1 and $-x_0$ of the notation of this paper (Sec. 33). In future the notation proper to this paper will be adhered to, and the variables of the binary quantic will be represented accordingly by x_1 and $-x_0$. Also, since

$$(a_0, a_1, \dots a_n \between x_1, -x_0)^n = (x_1 - ax_0)^n \, a_0,$$

the binary quantic will in general be expressed in the latter, and shorter, form.

76. In speaking of invariants it is not necessary always to refer to them as being invariants of a particular quantic or quantics, but, in accordance with the fact that they are fully represented by pure compound forms consisting of simple forms of marks a, b, c, &c., it will suffice to speak of them simply as invariants of marks a, b, c, &c., or as pure compound forms of those marks. The invariants of $(x_1 - ax_0)^n \, a_0$ may accordingly be spoken of as invariants of mark a; the quantic itself and its covariants being spoken of as invariants of marks a and x. (Sec. 39.)

VIII. *Seminvariants.*

77. The coefficient of a pure primary compound form of which the factor simple forms have respectively the marks a, b, c, ... and are respectively of valence a', b', c', ... is an invariant of the set of quantics

$$\begin{array}{l} (x_1 - ax_0)^{a'} \, a_0, \\ (x_1 - bx_0)^{b'} \, b_0, \\ (x_1 - cx_0)^{c'} \, c_0, \\ \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \end{array}$$

linear in the coefficients of each quantic (Sec. 35). If the marks a, b, c, ... are all replaced by the same mark p, then the coefficient of the pure primary compound form becomes an invariant I of the system of quantics

$$\begin{array}{l} (x_1 - px_0)^{a'} \, p_0, \\ (x_1 - px_0)^{b'} \, p_0, \\ (x_1 - px_0)^{c'} \, p_0, \\ \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \end{array}$$

where, of course, if any of the indices a', b', c', ... are the same, the quantics will be so also.

78. But this invariant I, besides being an invariant of the system of quantics, also bears a definite relation to a single quantic, viz., the quantic

$$(x_1-px_0)^n\,p_0,$$

where n is the greatest of the indices a', b', c', ... I is, in fact, a *seminvariant* of $(x_1-px_0)^n\,p_0$.

The proof is as follows. I may be expressed in the difference method as

$$I=(a-b)^{\lambda}(a-c)^{\mu}(b-c)^{\nu}\ldots,\quad a_0b_0c_0\ldots,\quad (a=b=c=\ldots=p),$$

and therefore $$\left(\frac{d}{da}+\frac{d}{db}+\frac{d}{dc}+\ldots\right)I=0.$$

But, since $$\frac{d}{da}a_r=\frac{d}{da}a^ra_0=ra^{r-1}a_0=ra_{r-1},$$

we may write

$$\frac{d}{da}=a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}+3a_2\frac{d}{da_3}+\ldots\ ad\ inf.,$$

and we have corresponding expressions for $\dfrac{d}{db}$, $\dfrac{d}{dc}$, &c. Hence, if we make

$$a=b=c=\ldots=p,$$

and therefore $$a_0=b_0=c_0=\ldots=p_0,$$

and $$a_1=b_1=c_1=\ldots=p_1,$$

and $$a_2=b_2=c_2=\ldots=p_2,$$

&c., &c.,

so that $$I=\phi\,(p_0,p_1,p_2\ldots p_n),$$

we have $$\left(p_0\frac{d}{dp_1}+2p_1\frac{d}{dp_2}+3p_2\frac{d}{dp_3}\ldots\right)I=0,$$

i.e., I is a seminvariant of $(x_1-px_0)^n\,p_0$.

It will be borne in mind that, since I is a seminvariant of $(x_1-px_0)^n\,p_0$, it is also a seminvariant of $(x_1-px_0)^{n+k}\,p_0$, where k is any positive integer; but, since I contains p_n, it will not be a seminvariant of any quantic $(x_1-px_0)^{n-k}\,p_0$.

79. It can also be shown that every seminvariant of $(x_1-px_0)^n p_0$ can be exhibited as the sum of one or more expressions, such as

$$(a-b)^\lambda (a-c)^\mu (b-c)^\nu \ldots a_0 b_0 c_0 \ldots \quad (a=b=c=\ldots=p),$$

and can therefore be represented as the coefficient of the sum of one or more pure compound forms. The proof is as follows.

Let S be any seminvariant of $(x_1-px_0)^n p_0$ of degree m. Then

$$\left(\Sigma_{r=\infty}^{r=0}\, rp_{r-1}\frac{d}{dp_r}\right) S = 0,$$

and therefore

$$\left(\Sigma a_r \frac{d}{dp_r}\right)\left(\Sigma b_r \frac{d}{dp_r}\right)\left(\Sigma c_r \frac{d}{dp_r}\right) \ldots \text{ to } m \text{ factors } \left(\Sigma rp_{r-1}\frac{d}{dp_r}\right) S = 0,$$

where the summation is in each case from $r=0$ to $r=\infty$. Now

$$\left(\Sigma q_r \frac{d}{dp_r}\right)\left(\Sigma rp_{r-1}\frac{d}{dp_r}\right) S = \left(\Sigma rq_{r-1}\frac{d}{dq_r}+\Sigma rp_{r-1}\frac{d}{dp_r}\right)\left(\Sigma q_r \frac{d}{dp_r}\right) S.$$

Hence we have (*)

$$\left(\Sigma a_r \frac{d}{dp_r}\right)\left(\Sigma b_r \frac{d}{dp_r}\right)\left(\Sigma c_r \frac{d}{dp_r}\right) \ldots \text{ to } m \text{ factors } \left(\Sigma rp_{r-1}\frac{d}{dp_r}\right) S$$

$$= \left(\Sigma ra_{r-1}\frac{d}{da_r}+\Sigma rb_{r-1}\frac{d}{db_r}+\Sigma rc_{r-1}\frac{d}{dc_r}+\ldots \text{ to } m \text{ terms}\right)$$

$$\times\left(\Sigma a_r \frac{d}{dp_r}\right)\left(\Sigma b_r \frac{d}{dp_r}\right)\left(\Sigma c_r \frac{d}{dp_r}\right) \ldots \text{ to } m \text{ factors } S.$$

But $\left(\Sigma a_r \frac{d}{dp_r}\right)\left(\Sigma b_r \frac{d}{dp_r}\right)\left(\Sigma c_r \frac{d}{dp_r}\right) \ldots$ to m factors $S = I$,

a function of the m sets of quantities

$$\begin{array}{llll} a_0, & a_1, & a_2, & a_3, \ldots, \\ b_0, & b_1, & b_2, & b_3, \ldots, \\ c_0, & c_1, & c_2, & c_3, \ldots, \\ \&c., & & \&c., & \end{array}$$

linear as to the quantities of each set. Hence, since $a_r = a^r a_0$, we may write

$$I = f(a, b, c, \ldots)\, a_0 b_0 c_0 \ldots,$$

and, since $\quad \Sigma r a_{r-1} \dfrac{d}{da_r} = \dfrac{d}{da}, \quad \Sigma r b_{r-1} \dfrac{d}{db_r} = \dfrac{d}{db}$, &c., &c.,

we have, by (*),

$$\left(\frac{d}{da} + \frac{d}{db} + \frac{d}{dc} \dots\right) f(a, b, c, \dots) = 0,$$

of which the solution is

$f(a, b, c, \dots) =$ the sum of terms such as $(a-b)^{\lambda}(a-c)^{\mu}(b-c)^{\nu}\dots$,

i.e. (Sec. 70), $I\Pi =$ the sum of one or more pure primary compound forms.

If now we put $\quad a = b = c = \dots = p,$

and therefore $\quad a_0 = b_0 = c_0 = \dots = p_0,$

and $\quad a_1 = b_1 = c_1 = \dots = p_1,$

and $\quad a_2 = b_2 = c_2 = \dots = p_2,$

&c., ... &c.,

I becomes $\underline{|m}\, S$, and each pure primary compound form becomes a pure degraded compound form, each factor simple form of which has the same mark p. Thus S will be represented by the sum of a number of seminvariants of $(x_1 - px_0)^n p_0$, each of which is a pure degraded compound form.

80. A seminvariant of the quantic $(x_1 - ax_0)^n a_0$ may be spoken of as a seminvariant "of mark a."

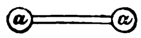
81. The coefficient of a pure compound form of which the factor simple forms are some of like and some of unlike marks, some of those of like marks being of different valence, will be a seminvariant of two or more quantics, and may be spoken of as a seminvariant of marks a, b, c, &c., where a, b, c, &c., are the marks of the component simple forms. It may be shown, by a proof similar to that in Sec. 79, that every seminvariant of two or more quantics can be represented by the coefficient of the sum of one or more pure compound forms containing simple forms of various marks and various valences.

82. The recognition of the fact that the seminvariants of a quantic $(x_1-ax_0)^n a_0$ are invariants of two or more quantics of the series

$$\begin{gathered} a_0, \\ (x_1-ax_0)\ a_0, \\ (x_1-ax_0)^2 a_0, \\ (x_1-ax_0)^3 a_0, \\ \cdots\ \cdots\ \cdots \\ (x_1-ax_0)^n a_0, \end{gathered}$$

or, as it may otherwise be expressed, that seminvariants of mark a differ from invariants of mark a, in this only, viz., that the simple forms which compose the pure compound forms representing the latter are all of the same valence, while those which compose the pure compound forms representing the former may differ from each other in valence, throws important light upon the structure of seminvariants and invariants. Thus we see that it is an essential feature of the seminvariant which is the source of the cubic-covariant J of $(x_1-ax_0)^3 a_0$ that the compound form , which represents it, consists of three simple forms of valence 1, 2, and 3 respectively, so that J is an invariant of the three quantics

$$\begin{aligned} (x_1-ax_0)\ a_0 &\equiv a_0x_1-a_1x_0, \\ (x_1-ax_0)^2 a_0 &\equiv a_0x_1^2-2a_1x_1x_0+a_2x_0^2, \\ (x_1-ax_0)^3 a_0 &\equiv a_0x^3-3a_1x_1x_0+3a_2x_1x_0^2-a_3x_0^3; \end{aligned}$$

while the source of the Hessian H, being represented by , is seen to consist essentially of two simple forms of the same valence 2, and is therefore an invariant of the single quantic

$$(x_1-ax_0)^2 a_0.$$

IX. *Breaking up of Pure Compound Forms.*

83. A pure compound form consists either of two or more pure compound forms unconnected by bonds, or of one continuous portion. By successive applications of the formulæ of Secs. 13 and 30, a continuous pure compound form may, without altering the valence of the factor simple forms, be expressed in various ways as the sum of other

pure compound forms, some or all of which may consist of two or more detached pure compound forms. If a continuous pure compound form is expressed as the sum of a number of pure compound forms, each of which thus consists of two or more detached portions, the continuous pure compound form may conveniently be said to be "broken up" into pure compound forms of lower degree and weight.

84. The coefficient of a pure compound form which consists of two or more distinct portions, will consist of two or more distinct coefficients multiplied together, and the expression of a pure compound form, as the sum of terms each of which consists of two or more detached pure compound forms, amounts to the expression of its coefficient as the sum of terms each of which is the product of two or more distinct coefficients, *i.e.*, amounts to the expression of an invariant as the sum of terms each of which is the product of two or more invariants.

85. From Gordan's theorem that the number of invariants of a quantic of given order, or a system of quantics of given orders, is finite, it follows immediately that a pure compound form containing simple forms of given valences and marks will always break up, if the number of such simple forms (*i.e.*, the degree of the pure compound form) is sufficiently large.

86. If we use the non-homogeneous formula of Sec. 63 (see also Sec. 71), we can increase the number of ways in which a pure compound form can be broken up; but, as the valence of individual factor simple forms will be altered in the process, the factor pure compound forms into which the original pure compound form breaks up, may, when algebraically expressed, contain letters which were not comprised in the algebraical expression of the original pure compound form; viz., if the original pure compound form contained a simple form of mark a and valence n, its algebraical expression would contain $a_0, a_1, a_2, \ldots a_n$, but not a_{n+1}, a_{n+2}, &c. If, however, in the process of breaking up, the valence of the simple form a becomes increased to $n+k$, we shall have $a_{n+1}, a_{n+2}, \ldots a_{n+k}$ in one or other of the pure compound forms into which the original form breaks up.

87. For example, to take a very simple case, the continuous form

$$\begin{aligned} F\Pi &= 2\,(a_1b_1c_0 - a_1b_0c_1 - a_0b_2c_0 + a_0b_1c_1)\,\Pi \\ &= 2\left\{(a-b)(b-c)\,a_0b_0c_0\right\}\Pi \\ &= \quad \left\{(a-c)^2\,a_0c_0\right\}\left\{b_0\right\}\Pi \\ &\quad -\left\{(a-b)^2\,a_0b_0\right\}\left\{c_0\right\}\Pi \\ &\quad -\left\{(b-c)^2\,b_0c_0\right\}\left\{a_0\right\}\Pi \\ &= \quad \left\{a_2c_0 - 2a_1c_1 + a_0c_2\right\}\left\{b_0\right\}\Pi \\ &\quad +\left\{2a_1b_1 - a_2b_0 - a_0b_2\right\}\left\{c_0\right\}\Pi \\ &\quad +\left\{2b_1c_1 - b_2c_0 - b_0c_2\right\}\left\{a_0\right\}\Pi\,; \end{aligned}$$

and thus $F\Pi$ breaks up, being expressed as the sum of three terms, each of which is the product of two wholly detached pure forms; but these forms contain the letters a_2 and c_2 which do not appear in the coefficient F.

88. It is not proposed to further discuss here the general question of the breaking up of pure compound forms. There is, however, an interesting and important theorem with regard to the breaking up of the product of any pure compound form and a special form of the same weight, which will be considered.

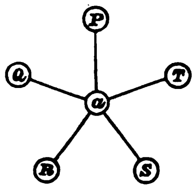
89. Let $a, b, c, \ldots$ and $P, Q, R, \ldots$ be the marks of the simple forms composing the two pure compound forms ϕ and Φ respectively. Then it is the immediate consequence of the identity

$$(a-b)(P-Q) = (a-P)(b-Q)-(a-Q)(b-P)$$

that, if ϕ and Φ are of the same weight w, so that the number of bonds in each is w, the product $\phi\Phi$ can be expressed as the sum of 2^w terms, each term being a pure compound form in which no bonds connect the simple forms $a, b, c, \ldots$ to each other, or the simple forms $P, Q, R \ldots$ to each other, but each bond connects one or other of the simple forms $a, b, c \ldots$ to one or other of the simple forms $P, Q, R, \ldots$.

90. Hence, if Φ be such that the simple forms $P, Q, R, \ldots$ composing it are all of valence 1, the product $\phi\Phi$ will be broken up into what may be called "star forms," each of which will consist of one of the simple forms $a, b, c, \ldots$ connected by radiating bonds to univalent terminal simple forms of the set P, Q, R, ..., the number of

such terminal simple forms being of course the same as the valence of the central simple form; *e.g.*, one of such star forms would be



this being represented in the difference method by

$$(a-P)(a-Q)(a-R)(a-S)(a-T)\, a_0 P_0 Q_0 R_0 S_0 T_0 \,.\, \Pi.$$

In order that the simple forms P, Q, R, ... may be all of valence 1, Φ must consist of w pairs of such simple forms connected by single bonds, *i.e.*, of w pairs, such as

$$\text{Ⓟ}\!\!-\!\!-\!\!-\!\!\text{Ⓠ} = (u)_P\,(u)_Q = (Q-P)\, P_0 Q_0\, UU'$$
$$= (P_0 Q_1 - P_1 Q_0)\, UU',$$

which may conveniently be termed "single bond forms." We have here the important result that the form resulting from the multiplication of any pure compound form ϕ, of weight w, by w single bond forms, can be broken up into star forms each containing one simple form only of those composing ϕ.

91. If the w single bond forms are all exactly alike, the two simple forms in each having the marks P and Q respectively, a star form σ containing the simple form a of ϕ may be expressed by

$$(a-A)(a-B)\ldots(a-P)(a-L)(a-M)\ldots(a-Q)\, a_0 A_0 B_0 \ldots$$
$$\ldots P_0 L_0 M_0 \ldots Q_0 \Pi \,.\, \begin{Bmatrix} A = B = \ldots = P \\ L = M = \ldots = Q \end{Bmatrix}$$
$$= (aA_0 - A_1)(aB_0 - B_1)\ldots$$
$$(aP_0 - P_1)(aL_0 - L_1)(aM_0 - M_1)\ldots(aQ_0 - Q_1)\, a_0 \Pi \qquad ,,$$
$$= (aP_0 - P_1)^{\lambda}\,(aQ_0 - Q_1)^{i-\lambda}\, a_0\, \Pi = (-1)^{i-\lambda}\,(aP_0 - P_1)^{\lambda}\,(Q_1 - aQ_0)^{i-\lambda}\, a_0 \Pi$$
$$= (-1)^{i-\lambda}\, \sigma_0 \Pi,$$

where i is the valence of the simple form a. We shall have such star forms for all values of λ from 0 to i.

92. If $P_0Q_1 - P_1Q_0 = \omega$, and ϕ_0 is the coefficient of ϕ, we have

$$\phi\Phi = \phi_0 \omega^w \Pi,$$

and the process we are considering shows how the invariant $\phi_0 \omega^w$ may be expressed as the sum of products of "star invariants," such as

$$(aP_0 - P_1)^\lambda (Q_1 - aQ_0)^{i-\lambda} a_0 = \sigma_0.$$

I proceed to consider some special cases. In all these I shall take $Q = \epsilon$ (Sec. 61), so that $Q_0 = 0$ and $Q_1 = 1$, and therefore $\omega = P_0$, and

$$\sigma_0 = (aP_0 - P_1)^\lambda a_0$$
$$= (a_0, a_1, a_2, \ldots a_\lambda \between -P_1, P_0)^\lambda.$$

93. First, then, let $P_0 = 1$, so that

$$\omega = 1, \quad \phi_0 \omega^w = \phi_0, \quad \text{and} \quad \sigma_0 = (a - P_1)^\lambda a_0.$$

Here the process under consideration shows how any invariant ϕ_0 may be expressed as a rational integral function of star invariants such as $(a - P_1)^\lambda a_0$, *i.e.*, invariants which are the coefficients of star forms consisting each of one or other of the simple forms contained in ϕ, with bonds radiating therefrom to univalent simple forms, each of which is one or other of the simple forms

$$(u)_\epsilon = U' \quad \text{or} \quad (u)_P = U + P_1 U';$$

or, more simply, since by Sec. 62 the simple forms such as $(u)_\epsilon$ may be omitted without affecting the value of the coefficient, we see that ϕ_0 may be expressed as a rational integral function of the coefficients of star forms which consist each of one or other of the simple forms of ϕ with bonds radiating to λ univalent forms of mark P, where $\lambda = 0, 1, 2, 3 \ldots i$, the valence of the simple form as a member of ϕ being i.

94. If we make $P_1 = 0$, since $P_0 = 1$, we have $P = \eta$ (Sec. 61), and the single bond form $(u)_P (u)_Q$ becomes the form

$(u)_\eta (u)_\epsilon = UU' =$  .

Also we have $\qquad \sigma_0 = a^\lambda a_0 = a_\lambda,$

and thus the coefficients of the star forms become simply

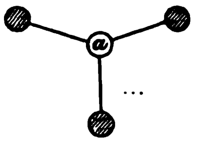
$$a_0, a_1, a_2, \ldots;\ b_0, b_1, b_2, \ldots;\ c_0, c_1, c_2, \ldots;\ \&c., \&c.,$$

viz., we have, omitting the univalent forms $(u)_e$,

$$= a_0,$$

$$= a_1 \Pi,$$

$$= a_2 \Pi,$$



$$= a_3 \Pi,$$

&c., &c.;

and the process gives us the invariant ϕ_0 in its usual algebraical form, viz., as a function of

$$a_0, a_1, a_2, \ldots;\ b_0, b_1, b_2, \ldots;\ c_0, c_1, c_2, \ldots;\ \text{\&c., \&c.}$$

95. Observe here that, the star forms of the last section being all pure compound forms, their coefficients are invariants; *e.g.*, a_k is an invariant, not however of a single quantic, but of the two quantics

$$(x_1 - ax_0)^\kappa a_0,$$

$$(x_1 - \eta x_0)\,\eta_0 = x_1,$$

since $$\eta_0 = 1 \quad \text{and} \quad \eta_1 = 0.$$

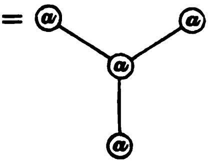
96. Next let all the simple forms of ϕ be of mark a, so that ϕ_0 is an invariant or seminvariant of mark a. Also let $P = a$, and therefore $P_0 = a_0$, and $P_1 = a_1$. Here $\omega = a_0$, and

$$\sigma_0 = (a_0 a - a_1)^\lambda a_0$$
$$= (a_0, a_1, a_2, \ldots a_\lambda)(-a_1, a_0)^\lambda,$$

and the process shows how the invariant or seminvariant ϕ_0 of mark a and weight w, when multiplied by a_0^w, may be expressed as a rational integral function of the quantities

$$\mathbf{a}_0 = a_0,$$
$$\mathbf{a}_1 = (a_0 a - a_1)\, a_0 = 0,$$
$$\mathbf{a}_2 = (a_0 a - a_1)^2 a_0 = a_0^2 a_2 - a_0 a_1^2,$$
$$\mathbf{a}_3 = (a_0 a - a_1)^3 a_0 = a_0^2 a_3 - 3a_0^2 a_2 a_1 + 2a_0 a_1^3,$$
&c., ... &c.

Here the star form $\mathrm{a}_\kappa \Pi$ consists [if we omit the forms of mark ϵ (Sec. 62)] of $\kappa+1$ simple forms of mark a, viz., one of valence κ and κ of valence 1, *e.g.*,

$\mathrm{a}_3 \Pi =$ 

Hence a_κ is an invariant of the two quantics

$$(x_1-ax_0)^\kappa a_0 \text{ and } (x_1-ax_0)\, a_0 = (a_0x_1-a_1x_0),$$

and is a seminvariant of the single quantic $(x_1-ax_0)^\kappa a_0$. The seminvariants a_0, a_2, a_3, ... &c. are those given by Professor Cayley in his paper "On Seminvariant Tables," *Amer. Jour. of Math.*, Vol. VII., p. 59.

97. Since $S_0\Pi$, where S_0 is any seminvariant of mark a and weight w, can be expressed as the sum of a number of pure compound forms such as ϕ_0 (Sec. 79), $S_0 a^w$ can be expressed as a rational integral function of a_0, a_2, a_3 ... .

98. Consider now the series of seminvariants $A_1 (=0)$, A_2, A_3 ... &c., where

$A_{p+1}\Pi =$  $= (a-b)^p(a-c)\, a_0b_0c_0\Pi.\ (a=b=c.)$

The compound form here, though of weight $p+1$, breaks up into star forms on multiplication by p of the single-bond forms $(u)_\epsilon\, (u)_\omega$, and we have

$$A_{p+1}a_0^p = \mathrm{a}_{p+1} + f(\mathrm{a}_p, \mathrm{a}_{p-1} \dots \mathrm{a}_0),$$

where f is a rational integral function. Hence

$$\mathrm{a}_{p+1} = A_{p+1}a_0^p - f(\mathrm{a}_p, \mathrm{a}_{p-1} \dots \mathrm{a}_0),$$

from whence it immediately follows that

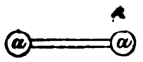
$$\mathrm{a}_{p+1} = F(A_{p+1}, A_p, A_{p-1} \dots A_2, a_0),$$

where F is also a rational integral function. Thus $S_0 a_0^w$ may be expressed as a rational integral function of the series of semin-

variants A_2, A_3, A_4 ... and a_0. Observe, however, that the degree and weight of A_3 are the same, viz., 3, and that for all values of κ greater than 3, the weight of A_κ, which is κ, exceeds the degree, which is 3. Also

$$2A_3 = 2\,(a-b)(a-c)\;a_0b_0c_0 \qquad (a=b=c.)$$
$$= (a-b)^2\,a_0b_0c_0 + (a-c)^2\,a_0b_0c_0 - (b-c)^2\,a_0b_0c_0 \qquad ,,$$
$$= (a-b)^2\,a_0b_0c_0 \qquad ,,$$
$$= a_0 . (a-b)^2\,a_0b_0 \qquad (a=b.)$$
$$= a_0 . A_2',$$

where A_2' is a seminvariant of equal degree and weight, viz.,

$$A_2'\Pi =$$  .

Hence $S_0a_0^w$ may be expressed as a rational integral function of a_0, A_2', A_3, A_4, ..., *i.e.*, of a_0 and a number of seminvariants of mark a, such that the degree of no one of them exceeds the weight. Now, if the degree of S_0 is j, the degree of $S_0a_0^w$ will be $w+j$, and its weight will be w; hence every term of the expression of $S_0a_0^w$ as a rational integral function of a_0, A_2', A_3, A_4 ..., &c., must contain a factor a_0^j. Dividing this out, we see that $S_0a_0^{w-j}$ can be expressed as a rational integral function of a_0, A_2', A_3, A_4, ...

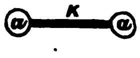
99. If κ be even, $= 2\kappa'$, we have

$$A_\kappa = A_{2\kappa'} = (a-b)^{2\kappa'-1}\,(a-c)\;a_0b_0c_0 \qquad (a=b=c.)$$
$$= (a-b)^{2\kappa'-1}\,(\overline{a-b}+\overline{b-c})\;a_0b_0c_0 \qquad ,,$$
$$= (a-b)^{2\kappa'}\,a_0b_0c_0 + (a-b)^{2\kappa'-1}\,(b-c)\;a_0b_0c_0 \qquad ,,$$
$$= (a-b)^{2\kappa'}\,a_0b_0c_0 - (a-b)^{2\kappa'-1}\,(a-c)\;a_0b_0c_0, \qquad ,,$$

by interchanging b and a in the second term (Sec. 74),

$$= a_0 . (a-b)^{2\kappa'}\,a_0b_0 - A_\kappa. \qquad (a=b.)$$

Hence $$2A_\kappa = a_0 . (a-b)^{2\kappa'}\,a_0b_0 \qquad (a=b.)$$
$$= a_0A_\kappa' \text{ say,}$$

where $$A_\kappa' =$$  .

We can therefore express $S_0 a_0^{w-j}$ as a rational integral function of the seminvariants $a_0, A_2', A_3, A_4', A_5, \dots$.*

X. *On the Relation of the Graphical Representations of Covariants and their Sources.*

100. Any compound form representing a covariant C of order g in the variables, and of degree j and weight w in the coefficients (*i.e.*, of total degree $g+j$ and weight $g+w$), will consist of g univalent simple forms of mark x, and j other simple forms representing the coefficients. If we multiply $C\Pi$ by g single-link forms, such as

 $= (u)_\epsilon\,(u)_\eta$,

the coefficients of which are 1, then, since

$$(a-x)(\epsilon-\eta) = (a-\epsilon)(x-\eta)-(a-\eta)(x-\epsilon),$$

we may express the product, the coefficient of which will also be C, as the sum of 2^g terms, each of which will consist of (1) a compound form $C_r\Pi$ precisely similar to $C\Pi$, but having univalent forms of marks ϵ and η (r of the former and $g-r$ of the latter), in *lieu* of those of mark x, and (2) of g single-bond forms, some (r) such as

 $= U\,(x_0 U + x_1 U') = x_1 U'U,$

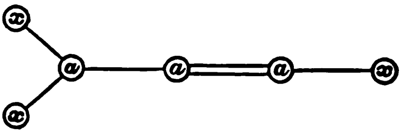
and the rest $(g-r)$ such as

 $= U'\,(x_0 U + x_1 U') = x_0 U'U;$

and thus we have a number of terms such as $C_r x_1^r x_0^{g-r}\,\Pi$. Here, since there are $\dfrac{\lfloor g}{\lfloor r\,\lfloor g-r}$ different ways in which we can substitute r univalent forms of mark ϵ, and $g-r$ of mark η for the g univalent forms of mark x, there will be $\dfrac{\lfloor g}{\lfloor r\,\lfloor g-r}$ terms such as $C_r x_1^r x_0^{g-r}\,\Pi$ for any value of r, which terms may all differ in value; and there will be terms for all values of r from 0 to g; in all, as before stated, 2^g terms. The sum of all the C_r's for any given value of r will be the coefficient of $x_1^r x_0^{g-r}$ in the expansion of the covariant C. In certain cases all the C_r's for any given value of r will be equal; *e.g.*, if $C\Pi$ contains merely

* See Sylvester "On Clebsch's Theory, &c.," *Amer. Jour. of Math.*, Vol. I., p. 118, and Hammond "On the Solution of the Differential Equation of Sources," *id.*, *Vol. v.*, *p. 218.*

one simple form of mark a besides those of mark x, so that C is the ordinary quantic $(x_1 - ax_0)^g a_0$, all the terms C_r will be equal, each being $= a_r$, in which case the coefficient of $x_1^r x_0^{g-r}$ is, apart from sign, $\frac{\lfloor g}{\lfloor r \, \lfloor g-r} a_r$. In every case the coefficient of x_1^g, *i.e.*, the leading term or source of the covariant C, will be represented by a single term $C_0\Pi$, all the univalent forms substituted for those of mark x being of mark ϵ. Since we may omit all simple forms of mark ϵ and their bonds (Sec. 62), without affecting the value of the coefficient of the compound form containing them, we see that the graphical representation of the source C_0 of any covariant C is obtained by omitting from the graphical representation of the covariant all the nuclei of mark x, and the bonds connecting them to the other nuclei. Thus the source of the cubic-covariant which is the coefficient of



is the coefficient of 
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On the Vibrations of an Elastic Circular Ring. (Abstract.)

By Mr. A. E. H. Love. Communicated January 12th, 1893.

The ring is supposed to be of small circular section of radius c, and the elastic central-line a circle of radius a. There are four ways of displacing the ring. A point on the central-line may move along the radius of the circle which is its primitive form, or perpendicularly to the plane of this circle, or along the tangent to this circle; and the circular sections may be displaced by rotation about the central-line. The modes of vibration fall into four classes, of which two are physically important.

Class I. *Flexural Vibrations in Plane of Ring.*—These were investigated by Hoppe in 1871 (*Crelle*, Bd. LXXIII.). The motion of a point on the elastic central-line is compounded of a displacement along the radius and a displacement along the tangent to the circle, so proportioned that the central-line remains unstretched and the nodes of the former displacement are the antinodes of the latter. There must be at least two wave-lengths to the circumference, and the frequency ($p/2\pi$) of the mode in which there are n wave-lengths to the circumference is given by the equation

$$p^2 = \frac{1}{4}\,\frac{n^2(n^2-1)^2}{1+n^2}\,\frac{E}{\rho_0}\,\frac{c^2}{a^4},$$

in which E is the Young's modulus, and ρ_0 the density of the material. Except for the numerical coefficient this is precisely similar to the formula for the lateral vibrations of a straight bar of the same material and section and of length πa. The sequence of component tones, when n is very great, is ultimately identical with that of the tones of a free-free bar of length πa, but the sequence for the low tones is quite different to that for a bar.

Class II. *Flexural Vibrations Perpendicular to the Plane of the Ring.* It is found to be impossible to make the ring vibrate freely so that each particle of the elastic central-line moves perpendicularly to the plane of the ring, unless at the same time the sections turn about the central-line through a certain angle. In other words the flexure perpendicular to the plane of the ring is always accompanied by *torsion.* As in Class I., there must be at least two wave-lengths to the circumference, and the frequency of the mode in which there are n wave-lengths to the circumference is given by the equation

$$p^2 = \frac{1}{4}\,\frac{n^2(n^2-1)^2}{1+\sigma+n^2}\,\frac{E}{\rho_0}\,\frac{c^2}{a^4},$$

where σ is the *Poisson's ratio* for the material, and the other constants have the same meaning as before. (For most hard solids σ is about $\frac{1}{4}$.) Since n must be at least 2, the sequence of tones is very nearly the same as in the vibrations of Class I., but the pitch is slightly lower, the ratio of the frequencies for the gravest tones being $\sqrt{\frac{21}{20}}$, which is very little greater than a *comma.* For the higher tones, as we should expect, there is no sensible difference.

These two classes include all that have much physical importance. The remaining types can be classified as follows :—

Class III. *Extensional Vibrations.*—The motion may be purely radial or partly radial and partly tangential. In the second case there will be an integral number of wave-lengths, and, when this number is n, we have the formula for the frequency

$$p^2 = (1+n^2)\frac{E}{\rho_0}\frac{1}{a^2}.$$

Putting $n = 0$, we find the frequency of the purely radial vibrations. The pitch of any mode of extensional vibration of the ring is of the same order of magnitude as the pitch of the corresponding longitudinal vibration of a bar of length equal to half the circumference, the formula for the latter being in fact derived by writing n^2 for $1+n^2$.

Class IV. *Torsional Vibrations.*—The motion consists of an angular displacement of the sections about the elastic central-line, accompanied by a relatively very small displacement of the points on this line perpendicular to the plane of the ring. When there are n wave-lengths to the circumference, the frequency is given by the formula

$$p^2 = (1+\sigma+n^2)\frac{\mu}{\rho_0}\frac{1}{a^2},$$

in which μ is the *rigidity* of the material. There is one symmetrical mode for which n is zero, and, since $2\mu(1+\sigma) = E$, the frequency of this mode is $\frac{1}{2}\sqrt{2}$ of that of the radial vibrations. The pitch of the torsional vibrations is comparable with that for a straight rod of length equal to half the circumference, the formula for the latter being in fact derived by writing n^2 in place of $1+\sigma+n^2$. Formulæ equivalent to those given in connexion with Classes II. and IV. have been obtained by Mr. Basset (*Proc.*, December, 1891), but he has not interpreted his results.

Note on Secondary Tucker-Circles. By JOHN GRIFFITHS, M.A.
Read November 10th and December 8th, 1892. Received in revised form February 4th, 1893.

The principal theorems discussed in this paper are particular cases of the following proposition, viz., If DEF denote a triangle of given species having its vertices D, E, F respectively on the sides BC, CA, AB of a given triangle ABC, or on these sides produced, then DEF will belong to one or other of a pair of systems of similar inscribed triangles, and each of these systems will have a common centre of similitude. In fact, if D, E, F be the given angles of the inscribed triangle DEF, there will be a primary system of similar inscribed triangles whose common centre of similitude is the point given by the isogonal coordinates

$$x = \frac{\sin(D+A)}{\sin D}, \quad y = \frac{\sin(E+B)}{\sin E}, \quad z = \frac{\sin(F+C)}{\sin F},$$

and also a secondary system whose centre of similitude is the point represented by

$$x = \frac{\sin(D-A)}{\sin D}, \quad y = \frac{\sin(E-B)}{\sin E}, \quad z = \frac{\sin(F-C)}{\sin F}.$$

These two centres of similitude are the inverse of each other with respect to the circumcircle ABC. For example, if $D = A$, $E = B$, $F = C$, then the centre of similitude of the primary system of inscribed triangles will be the point $(2\cos A, 2\cos B, 2\cos C)$, *i.e.*, the centre of the circumcircle ABC, while that of the secondary system will be an infinitely distant point $(0, 0, 0)$.

Moreover, the circle circumscribing any triangle DEF of the first system will have double contact with the inscribed conic

$$\Sigma \sqrt{x \sin D \sin(D+A)} = 0,$$

and the corresponding conic for the circumcircle of a triangle of the second system will be represented by

$$\Sigma \sqrt{x \sin D \sin(D-A)} = 0.$$

The particular examples considered in the note are the primary and secondary systems of inscribed triangles corresponding to the

values (1) $\angle D = B$, $\angle E = C$, $\angle F = A$; (2) $\angle D = C$, $\angle E = A$, $\angle F = B$.

For the primary systems of in-triangles under consideration, the centres of similitude are the Brocard points, and the corresponding circumscribed circles are well-known as Tucker-circles.

The secondary systems, whose centres of similitude are the inverse with respect to the circumcircle ABC of the Brocard points, have not hitherto, so far as I know, been noticed.

In each of these we have a series of triangles directly similar to each other, but—unlike the Tucker-triangles—inversely similar to the triangle of reference ABC.

Section I.

The results arrived at will be more readily understood by an explanation regarding what I have called isogonal coordinates, which can be employed to investigate the properties of systems of circles connected with the triangle (see my "Notes on the Recent Geometry of the Triangle").

Briefly, if α, β, γ be the trilinear coordinates of a point G, in the plane of a given triangle of reference ABC, the isogonal coordinates of G are given by

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma} = \frac{a\alpha + b\beta + c\gamma}{a\beta\gamma + b\gamma\alpha + c\alpha\beta},$$

where a, b, c denote the sides BC, CA, AB.

It is thus easily seen that these coordinates x, y, z satisfy the relation

$$ax + by + cz = ayz + bzx + cxy, \quad \text{or} \quad \Sigma x \sin A = \Sigma yz \sin A,$$

which is unaltered by writing therein x^{-1}, y^{-1}, z^{-1} for x, y, z.

Again, if we take a point G inside the circumcircle ABC, and draw perpendiculars GD, GE, GF from it to the sides BC, CA, AB, it may be proved that the isogonal coordinates of G are

$$x = \frac{\sin(D+A)}{\sin D}, \quad y = \frac{\sin(E+B)}{\sin E}, \quad z = \frac{\sin(F+C)}{\sin F},$$

where D, E, F are the angles of the pedal triangle DEF. These coordinates satisfy the relation

$$\Sigma x \sin A = \Sigma yz \sin A.$$

For example, let G coincide with O, the centre of the circumcircle ABC, then $D = A$, $E = B$, $F = C$, so that

$$x = 2\cos A, \quad y = 2\cos B, \quad z = 2\cos C$$

are the isogonal coordinates of the centre of the circumcircle ABC.

If G be taken outside the circumcircle ABC, the isogonal coordinates of G, in terms of the angles D, E, F of its pedal triangle DEF, are

$$x = \frac{\sin(D-A)}{\sin D}, \quad y = \frac{\sin(E-B)}{\sin E}, \quad \text{and} \quad z = \frac{\sin(F-C)}{\sin F}.$$

Supposing then we have two points, G and g, whose isogonal coordinates are respectively

$$x = \frac{\sin(D+A)}{\sin D}, \quad x' = \frac{\sin(D-A)}{\sin D}, \ \&c.,$$

it follows that

$$x+x' = 2\cos A, \quad y+y' = 2\cos B, \quad z+z' = 2\cos C.$$

G and g will then, in fact, be a pair of points inverse to each other with respect to the circumcircle ABC, so that

$$OG \,.\, Og = R^2,$$

where O denotes the centre and R the radius of the circle ABC.

Conversely, if G, g be two inverse points with respect to the circumcircle ABC, then the pedal triangles of these points are similar. This result is an important one in the geometry of the triangle.

The following theorem, for example, is deduced at once from it. If the cotangents of the angles of the pedal triangle DEF, with respect to ABC, of a point P, be connected by a linear relation

$$\Sigma\lambda \cot D = \text{const.},$$

then the locus of P will, in general, be a pair of circles inverse to each other with respect to the circumcircle ABC. As a particular case, it follows that, if P be any point either on the Brocard circle

$$\Sigma x \operatorname{cosec} A = 2\cot\omega$$

or on its inverse line $\quad \Sigma x \operatorname{cosec} A = 0,$

the pedal triangle of P with respect to ABC has the same Brocard angle as ABC.

Again, the pedal triangle DEF of a point G can be turned in its own plane round G as a fixed centre of similitude, with its vertices

D, E, F moving on the sides BC, CA, AB of the triangle of reference. The angles D, E, F will, in fact, remain constant, and the circle DEF will have double contact with an inscribed conic which has $G(x, y, z)$ for a focus, the chord of contact being parallel to the transverse axis of the conic. The equation of this curve may be written in various forms, one of which is

$$\Sigma \sqrt{ax(bz+cy-a)\alpha} = 0.$$

For example, let G coincide with the negative Brocard point $\left(\frac{b}{c}, \frac{c}{a}, \frac{a}{b}\right)$; here

$$x = \frac{\sin(C+A)}{\sin C}, \quad y = \frac{\sin(A+B)}{\sin A}, \quad z = \frac{\sin(B+C)}{\sin B},$$

so that, for the pedal triangle of the point, we have

$$\angle D = C, \quad \angle E = A, \quad \angle F = B,$$

and, for the equation of the inscribed conic in question,

$$\Sigma \sqrt{bc\alpha} = 0.$$

This curve is known as the Brocard ellipse, and the circle DEF is a Tucker-circle.

SECTION 2. *Secondary Tucker-Circles.*

As a particular case of the above general theorem, viz., that the pedal triangles of a pair of inverse points with respect to the circum-circle ABC are similar, I consider the systems of circles corresponding to the Brocard points and their inverse points as here defined.

1. Taking the negative Brocard point $\left(\frac{b}{c}, \frac{c}{a}, \frac{a}{b}\right)$, we have seen that the angles of its pedal triangle are $D = C$, $E = A$, $F = B$, and, if the triangle be turned round this point in the manner explained in Section 1, the Tucker-circle DEF will have double contact with the Brocard ellipse

$$\Sigma \sqrt{bc\alpha} = 0.$$

Again, the pedal triangle def of the inverse point g will be similar to the corresponding Tucker triangle DEF, or $\angle d = C$, $\angle e = A$, $\angle f = B$, and if the triangle def be turned round g as above, the circle

def will have double contact with the conic

$$\Sigma \sqrt{a \sin C \sin (C-A)} = 0.$$

The coordinates of g are, in fact,

$$x = 2 \cos A - \frac{b}{c} = \frac{\sin (C-A)}{\sin C}, \quad y = 2 \cos B - \frac{c}{a} = \frac{\sin (A-B)}{\sin A},$$

and
$$z = 2 \cos C - \frac{a}{b} = \frac{\sin (B-C)}{\sin B},$$

so that the equation $\Sigma \sqrt{ax\,(bz+cy-a)\,a} = 0$

becomes
$$\Sigma \sqrt{a \sin C \sin (C-A)} = 0.$$

I propose to call the system of circles corresponding to the point inverse to a Brocard point a secondary Tucker system.

As I have just explained, if the Brocard point be $\left(\frac{b}{c}, \frac{c}{a}, \frac{a}{b}\right)$, the Tucker inscribed triangle DEF and the corresponding triangle *def* are similar; also the Tucker-circle DEF has double contact with the conic

$$\Sigma \sqrt{a \sin B \sin C} = 0,$$

and the secondary circle *def* with the conic

$$\Sigma \sqrt{a \sin C \sin (C-A)} = 0.$$

It may be here observed that one form of the equation of a circle *def* of this secondary system is

$$k_1 (1+k_2) x_1 x + k_2 (1+k_3) y_1 y + k_3 (1+k_1) z_1 z + 1 = 0,$$

where $x_1 = \dfrac{\sin (C-A)}{\sin C}, \quad y_1 = \dfrac{\sin (A-B)}{\sin A}, \quad z_1 = \dfrac{\sin (B-C)}{\sin A},$

and k_1, k_2, k_3 are connected by the relations

$$k_1 + k_2 + k_3 + 2 = 0, \quad a^2 (1+k_2) + b^2 (1+k_3) + c^2 (1+k_1) = 0.$$

2. In a similar manner, if we take G to coincide with the positive Brocard point $\left(\frac{c}{b}, \frac{a}{c}, \frac{b}{a}\right)$, the inverse point g will be given by the coordinates

$$x = 2 \cos A - \frac{c}{b}, \quad y = 2 \cos B - \frac{a}{c}, \quad z = 2 \cos C - \frac{b}{a},$$

or $$x = \frac{\sin(B-A)}{\sin B},\quad y = \frac{\sin(C-B)}{\sin C},\quad z = \frac{\sin(A-C)}{\sin A};$$

also in the triangles DEF, def the angles will be

$$D = d = B,\quad E = e = C,\quad \text{and}\quad F = f = A.$$

Lastly, the Tucker-circle DEF will have double contact with the Brocard ellipse

$$\Sigma\sqrt{a\sin B\sin C} = 0,$$

and the secondary circle def with the conic

$$\Sigma\sqrt{a\sin B\sin(B-A)} = 0.$$

As an example of a secondary circle, illustrating the difference between it and an ordinary Tucker-circle, I notice the following, viz.:— Let the tangent at A to the circumcircle ABC meet BC produced in d; through d draw de parallel to AB and meeting AC produced in e; then the circle Aed is a secondary Tucker-circle. Here the point f coincides with A, and the angles of the triangle Aed or fed are $d = C$, $e = A$, $f = B$. Also, if the other points where this circle meets the lines BC, CA, AB be denoted by d', e', f', then e' coincides with A, and the angles of the triangle $d'e'f'$ are $d' = A-C$, $e' = 180° - (A-B)$, $f' = C-B$, supposing $A > C > B$. If the circle were an ordinary Tucker-circle, we could have $d = C$, $e = A$, $f = B$, as before, but for the triangle $d'e'f'$ the angles would be $d' = B$, $e' = C$, $f' = A$.

In the case of the secondary circle in question, the equation of the double-contact inscribed conic is

$$\Sigma\sqrt{a\sin C\sin(C-A)} = 0,$$

whereas for an ordinary Tucker-circle it would be

$$\Sigma\sqrt{a\sin C\sin(C+A)} = 0.$$

It may be remarked that two of the foci of the curves

$$\Sigma\sqrt{a\sin B\sin(B-A)} = 0$$

and $$\Sigma\sqrt{a\sin C\sin(C-A)} = 0,$$

viz., the points $$\left(\frac{\sin(B-A)}{\sin B},\ \frac{\sin(C-B)}{\sin C},\ \frac{\sin(A-C)}{\sin A}\right)$$

and $$\left(\frac{\sin(C-A)}{\sin C},\ \frac{\sin(A-B)}{\sin A},\ \frac{\sin(B-C)}{\sin B}\right),$$

lie on the circle $\Sigma\,(b^2c^2-a^4)\,bcx + 3a^2b^2c^2 - \Sigma a^6 = 0$,

which touches the Brocard circle at the centre of the circumcircle (ABC). The first-mentioned circle is the inverse with respect to (ABC) of the line joining the Brocard points, since the expression $\Sigma\,(b^2c^2-a^4)\,bcx$ is transformed into $a^6+b^6+c^6-3a^2b^2c^2-\Sigma\,(b^2c^2-a^4)\,bcx$ by writing therein $2\cos A-x$, $2\cos B-y$, $2\cos C-z$ for x, y, z.

SECTION 3.

I here notice briefly some additional properties connected with the above pair of secondary systems of similar in-triangles, and the double-contact conics of the circumcircles.

1. If $d = B$, $e = C$, $f = A$, the sides de, ef, fd of the in-triangle def are parallel respectively to lines AP, BP, CP, which meet in a point P on the circumcircle ABC. See Fig. 1.

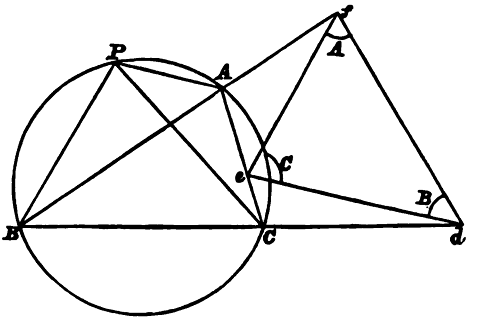


FIG. 1.

2. In like manner, when $d = C$, $e = A$, $f = B$, the sides de, ef, fd are parallel to lines BP, CP, AP, which also meet in P on the circumcircle. See Fig. 2.

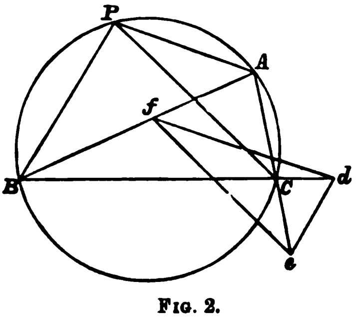


FIG. 2.

3. If a_1, b_1, e_1; a_2, b_2, e_2 denote the semi-axes and eccentricities of the above double-contact conics, and ω be the Brocard angle of the triangle ABC, then

$$a_1^2 = a_2^2 = R^2 \frac{\sin^2\omega}{1-4\sin^2\omega},$$

where R = radius of circle ABC,

$$-b_1^2 = b_2^2 = 4R^2 \frac{\sin^4\omega}{(1-4\sin^2\omega)^2} \frac{\sin(B-C)\sin(C-A)\sin(A-B)}{\sin A \sin B \sin C},$$

$$e_1^2 - 1 = 1 - e_2^2 = \frac{4\sin^2\omega}{1-4\sin^2\omega} \frac{\sin(B-C)\sin(C-A)\sin(A-B)}{\sin A \sin B \sin C},$$

or $$a_1 = a_2, \quad b_1^2 + b_2^2 = 0, \quad e_1^2 + e_2^2 = 2.$$

It thus follows that one of the conics is an ellipse and the other a hyperbola.

4. If a triangle $A'B'C'$ be inscribed in the circumcircle ABC, and circumscribed to the Brocard ellipse, it is known that $A'B'C'$ has the same Brocard angle ω and the same Brocard points as ABC. Hence, by means of the formulæ for the secondary double-contact conics, given above, I have deduced the following theorem, viz.:—

One focus of each of the secondary double-contact conics corresponding to a triangle $A'B'C'$—as defined above—remains fixed if the triangle of reference ABC be fixed, and the other two foci lie respectively on one of two fixed circles which have a common radius $2R \sin\omega$, and whose centres coincide with the fixed Brocard points of ABC.

5. If $A = \frac{\pi}{7}$, $B = \frac{2\pi}{7}$, and $C = \frac{4\pi}{7}$, one of the secondary double-contact conics is a circle, and the other an equilateral hyperbola. In this case one of the secondary Tucker systems consists of a system of concentric circles.

6. The inverse of the Brocard points of a triangle ABC, with respect to the circle ABC, are the Brocard points of a triangle similar and similarly placed to ABC, the centre of similitude being the centre of the circle ABC. If R', R denote the radii of the circumcircles of these two triangles, then

$$R' = R \div \sqrt{1-4\sin^2\omega}.$$

[The following is added in answer to questions suggested by one of the referees, viz.: What is the centre of similitude of a given triangle ABC, and an in-triangle inversely similar to it; and what is the locus of this point as the latter triangle moves?

So far as I have studied the problem, I have arrived at the following results with regard to centre of similitude (S, say) of ABC, and an inversely similar escribed triangle def, where $d = B$, $e = C$, $f = A$. See Fig. 1.

1. The point S is given by the trilinear equations

$$(1+k_2)\,a\alpha = (1+k_3)\,b\beta = (1+k_1)\,c\gamma,$$

where k_1, k_2, k_3 are connected by the relations

$$k_1+k_2+k_3+1 = 0 \quad \text{and} \quad \Sigma a^2 k_1 = 0.$$

2. The locus of S is the circumscribed conic represented by

$$\Sigma\beta\gamma \sin C \cos B = 0, \quad \text{or} \quad \Sigma yz \sin C \cos B = 0.$$

This curve passes through the point

$$x = \frac{\sin C}{\sin (C-A)}, \quad y = \frac{\sin A}{\sin (A-B)}, \quad z = \frac{\sin B}{\sin (B-C)},$$

i.e., through the isogonal conjugate with respect to the triangle ABC of the following point, viz., the inverse with regard to the circumcircle ABC of the negative Brocard point of ABC. The point in question is a focus of one of the double-contact conics discussed in the note.

3. The point P, on the circumcircle ABC, in which the lines AP, BP, CP, parallel to sides de, ef, fd, meet, is given by

$$k_1 a\alpha' = k_2 b\beta' = k_3 c\gamma',$$

where k_1, k_2, k_3 are subject to the same relations as before, viz.,

$$1+\Sigma k_1 = 0 \quad \text{and} \quad \Sigma a^2 k_1 = 0.$$

It thus appears that there is a correspondence between P and the centre of similitude S, which is expressed by the equations

$$\left(\frac{1}{a\alpha'}+\frac{1}{c\gamma'}\right) a\alpha = \left(\frac{1}{b\beta'}+\frac{1}{a\alpha'}\right) b\beta = \left(\frac{1}{c\gamma'}+\frac{1}{b\beta'}\right) c\gamma.$$

4. Again, the referee suggests the following question:—Let ABC be a given triangle, $A'B'C'$ an inversely similar escribed triangle.

Then, as $A'B'C'$ moves, its centre of similitude (U, say) is the inverse of a Brocard point of ABC. What relation has U to $A'B'C'$?

If U be taken to be the inverse, with respect to the circumcircle ABC, of the positive Brocard point of ABC, I have found that the isogonal coordinates of U, with reference to $A'B'C'$, are

$$x' = \frac{\sin C'}{\sin(C'-A')}, \quad y' = \frac{\sin A'}{\sin(A'-B')}, \quad z' = \frac{\sin B'}{\sin(B'-C')}.$$

It follows, then, that U is the isogonal conjugate, with reference to $A'B'C'$, of the following point, viz., the inverse, with respect to the circumcircle $A'B'C'$, of the negative Brocard point of $A'B'C'$.

The position of U with reference to $A'B'C'$ may be determined by considering the angles subtended at it by the sides $B'C'$, $C'A'$, $A'B'$. These are C', $\pi-A'$, B'; C', A', $\pi-B'$; $\pi-C'$, A', B', according as A', B' or C' is the least angle.

These results have been deduced by means of the following theorem, viz.: If the angles subtended by the sides BC, CA, AB of a triangle ABC, at a point G situated outside ABC, be $\pi-\theta$, ϕ, ψ; θ, $\pi-\phi$, ψ; or θ, ϕ, $\pi-\psi$, the isogonal coordinates of G will be

$$x = \frac{\sin\theta}{\sin(\theta-A)}, \quad y = \frac{\sin\phi}{\sin(\phi-B)}, \quad z = \frac{\sin\psi}{\sin(\psi-C)}.$$

Similarly, if $\pi-\theta$, $\pi-\phi$, $\pi-\psi$ denote the angles in question for an internal point G, then the isogonal coordinates of G with reference to ABC will be

$$x = \frac{\sin\theta}{\sin(\theta+A)}, \quad y = \frac{\sin\phi}{\sin(\phi+B)}, \quad \text{and} \quad z = \frac{\sin\psi}{\sin(\psi+C)}.$$

In all cases the angles θ, ϕ, ψ are the angles of the pedal triangle, with respect to ABC, of the following point, viz., the isogonal conjugate of G with reference to ABC.

For example, the angles subtended at the centre of the circumcircle of an acute-angled triangle ABC by the sides BC, CA, AB are $2A$, $2B$, $2C$, so that $\theta = \pi-2A$, $\phi = \pi-2B$, $\psi = \pi-2C$ are the angles of the pedal triangle of the orthocentre of ABC. The centre of the circumcircle and the orthocentre are, of course, isogonal conjugates.]

[redacted]

On a Group of Triangles Inscribed in a Given Triangle ABC, whose sides are Parallel to Connectors of any Point P with A, B, C. By R. Tucker, M.A. Received September, 1892. Read November 10th, 1892. Revised February, 1893.

1. Let DEF be one of the in-triangles, having its sides DE, EF FD respectively parallel to BP, CP, AP, and suppose

$$BD = pa, \quad CD = qa, \quad p+q \equiv 1.$$

If the trilinear coordinates of P are λ, μ, ν, the equations to DF, DE are

$$\begin{vmatrix} \alpha & \beta & \gamma \\ 0 & qc & pb \\ \begin{vmatrix} \gamma & -\mu \\ b & c \end{vmatrix} & \begin{vmatrix} -\mu & 0 \\ c & a \end{vmatrix} & \begin{vmatrix} 0 & \nu \\ a & b \end{vmatrix} \end{vmatrix} = 0,$$

$$\begin{vmatrix} \alpha & \beta & \gamma \\ 0 & qc & pb \\ \begin{vmatrix} 0 & -\lambda \\ b & c \end{vmatrix} & \begin{vmatrix} -\lambda & \nu \\ c & a \end{vmatrix} & \begin{vmatrix} \nu & 0 \\ a & b \end{vmatrix} \end{vmatrix} = 0;$$

i.e.,
$$\alpha a\,(bp\mu - cq\nu) + pb\beta u - qc\gamma u = 0,$$

$$\alpha\,(c\nu + pa\lambda) + pb\beta\lambda - qc\gamma\lambda = 0.$$

For shortness, write

$$u = b\mu + c\nu, \quad v = c\nu + a\lambda, \quad w = a\lambda + b\mu,$$

$$u+v+w = 2\Sigma_1, \quad a\lambda u + b\mu v + c\nu w = 2\Sigma_2.$$

From the above equations, after reduction, we obtain the equation to EF to be

$$-\alpha a\,(c\nu + pa\lambda)(cq\nu - bp\mu) + pb\beta u\,(c\nu + pa\lambda) + acq\lambda\gamma\,(cq\nu - bp\mu) = 0,$$

and this is parallel to $\alpha\mu - \beta\lambda = 0$,

i.e., CP; therefore

$$\frac{p}{ca\nu\lambda} = \frac{q}{b\mu v} = \frac{1}{\Sigma_2}.$$

Writing for p, q their values in the above equations, we get the equations to DE, EF, FD to be

$$\left.\begin{aligned} \nu vw\alpha + ab\lambda^2\nu\beta - b\lambda\mu v\gamma &= 0 \\ -c\mu\nu w\alpha + \lambda wu\beta + bc\mu^2\lambda\gamma &= 0 \\ ca\nu^2\mu\alpha - a\nu\lambda u\beta + \mu uv\gamma &= 0 \end{aligned}\right\} \quad \text{(i.)}.$$

2. In a similar manner we find the equations to $D'E'$, $E'F'$, $F'D'$, the sides of the other in-triangle, to be

$$\left.\begin{aligned} ab\mu^2\nu\alpha + \nu wu\beta - a\lambda\mu u\gamma &= 0 \\ -b\mu\nu v\alpha + bc\nu^2\lambda\beta + \lambda uv\gamma &= 0 \\ \mu vw\alpha - c\nu\lambda w\beta + ca\lambda^2\mu\gamma &= 0 \end{aligned}\right\} \quad \text{(ii.)}.$$

3. From (i.), (ii.), we can write the coordinates of the angular points

$$\left\|\begin{matrix} D, & 0, & \mu v, & a\nu\lambda \\ E, & b\lambda\mu, & 0, & \nu w \\ F, & \lambda u, & c\mu\nu, & 0 \end{matrix}\right\| ; \quad \left|\begin{matrix} D', & 0, & a\lambda\mu, & \nu w \\ E', & \lambda u, & 0, & b\mu\nu \\ F', & c\nu\lambda, & \mu v, & 0 \end{matrix}\right| , \text{ (modulus } \Sigma_2) \ \ldots \text{(iii.)}.$$

4. If L, M, N, are the orthogonal projections of P on BC, CA, AB, we have

$$\left.\begin{matrix} L, & 0, & \mu + \lambda\cos C, & \lambda\cos B + \nu \\ M, & \mu\cos C + \lambda, & 0, & \nu + \mu\cos A \\ N, & \lambda + \nu\cos B, & \nu\cos A + \mu, & 0 \end{matrix}\right| , \text{ (modulus } \Sigma_1) \ \ldots \text{(iv.)}.$$

The equation to the circle LMN is

$$2\Delta\,\Sigma(a\mu\nu)\,\Sigma(\alpha\beta\gamma) = \Sigma(a\alpha)\,\Sigma\left\{\lambda bc\,.\,\nu + \mu\cos A\,.\,\mu + \nu\cos A\,.\,\alpha\right\} \ldots \text{(v.)}.$$

5. From (iii.), we obtain

$$DD' = BD' \sim BD = a(c\nu w - ca\nu\lambda)/\Sigma_2 = abc\mu\nu/\Sigma_2 = FE' \ldots\ldots \text{(vi.)},$$

$$FF' = D'E = abc\lambda\mu/\Sigma_2,$$

and $DE' = abc\left\{\Sigma(\lambda^2\mu^2) + 2\lambda\mu\nu(\lambda\cos A + \mu\cos B - \nu\cos C)\right\}^{\frac{1}{2}}/\Sigma_2$.

Hence the perimeter of the hexagon

$$DD'EE'FF' = 2abc\,\Sigma(\lambda\mu)/\Sigma_2.$$

When P is the in-centre, this hexagon is equilateral, each side $= abc/\Sigma(ab)$. *Cf. Educational Times*, October, 1892, Quest. 11706.

6. From (i.) and (ii.), we see that DE, $D'F'$ intersect in

$$\frac{\beta}{\mu}=\frac{\gamma}{\nu}=\frac{avw}{bc\lambda\mu\nu},\quad i.e., \text{ on } AP \quad \ldots\ldots\ldots\ldots\ldots(\text{vii.}),$$

and EF, $E'F'$ in a_1 (say), given by

$$\frac{b\beta}{v}=\frac{c\gamma}{w}=\frac{bc\mu\nu a}{\lambda\,(b^2\mu^2+c^2\nu^2+\Sigma_2)};$$

hence Aa_1, Bb_1, Cc_1, where b_1, c_1, are analogous points to a_1, cointersect in P', *i.e.*,

$$a\alpha/u = b\beta/v = c\gamma/w \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{viii.}).$$

If P is the orthocentre, P' is the circumcentre;
" circumcentre, " nine-point centre;
" symmedian point, " isotomic of the inverse of the centre of the Brocard ellipse;

and if P' is the symmedian point, then P is the point

$$a^2\alpha \sec A = b^2\beta \sec B = c^2\gamma \sec C.$$

If P is the centroid, P' evidently coincides with P. In the case of P being one of the cosine-orthocentres (see *Messenger of Mathematics*, No. 199, p. 100), P' is on $\sigma_1 G$, or on $\sigma_2 G$.

7. The equation to PP' is

$$a\alpha\,(b\mu-c\nu)+\ldots+\ldots=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{ix.}),$$

which evidently passes through the centroid (G), and, further, G divides PP' so that

$$PG = 2P'G.$$

8. The equations to BE', CF' are

$$a/\lambda u = \gamma/b\mu\nu, \quad a/\lambda u = \beta/c\nu\mu;$$

hence they intersect on the median through A ....(x.).

9. Again, the equations to BE, CF' are

$$a/b\mu\lambda = \gamma/\nu w, \quad a/c\nu\lambda = \beta/\mu v;$$

hence, if these intersect in a_2, CF, AD' in b_2, AD, BE' in c_2, then Aa_2, Bb_2, Cc_2 cointersect in P_1, given by

$$a/a\lambda^2 u = \beta/b\mu^2 v = \gamma/c\nu^2 w \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{xi.}),$$

which is a point on PP'.

10. If a', b', c' are the lengths of the sides of LMN, we have

$$\left.\begin{aligned} a'^2 &= \mu^2+\nu^2+2\mu\nu\cos A,\\ b'^2 &= \nu^2+\lambda^2+2\nu\lambda\cos B,\\ c'^2 &= \lambda^2+\mu^2+2\lambda\mu\cos C.\end{aligned}\right\}$$

Now $\qquad BD = ca^2\lambda\nu/\Sigma_2,\quad BF = ca\lambda u/\Sigma_2;$

hence $\qquad DF'^2 = a^2b^2c^2\lambda^2(\mu^2+\nu^2+2\mu\nu\cos A)/\Sigma_2^2,$

and therefore

$$\left.\begin{aligned} DF &= abc\lambda a'/\Sigma_2 = D'E'\\ DE &= abc\mu b'/\Sigma_2 = E'F'\\ EF &= abc\nu c'/\Sigma_2 = D'F'\end{aligned}\right\}\text{..........................(xii.).}$$

The $\qquad$ triangle $DCE = \Delta\,.\,ab^2\lambda\mu^2 v/\Sigma_2^2;$

hence

$$\text{triangle } DEF = \Delta\left\{1-\frac{a\lambda(b\mu)^2v+b\mu(c\nu)^2w+c\nu(a\lambda)^2u}{\Sigma_2^2}\right\}$$

$$= \Delta abc\lambda\mu\nu\Sigma_1/\Sigma_2^2 = \Delta D'E'F'\text{..........................(xiii.).}$$

11. The equations to DE', $D'F$ respectively are

$$\left.\begin{aligned} b\mu^2\nu v\alpha+a\nu\lambda^2u\beta-\lambda\mu u v\gamma &= 0\\ c\nu^2\mu w\alpha-\nu\lambda w u\beta+a\lambda^2\mu u\gamma &= 0\end{aligned}\right\}\text{......................(xiv.);}$$

these intersect in P'', *i.e.*,

$$\alpha/\lambda u = \beta/\mu v = \gamma/\nu w,\quad\text{(modulus } 2\Sigma_2).$$

This point is obviously the centre of perspective of the pair of triangles, and also the centre of the conic through their six vertices.

If P is the circumcentre, P'' is the point $a\cos A\cos(B-C)$, ..., ...;

,,	orthocentre,	,,	symmedian point;
,,	symmedian point,	,,	centre of the Brocard ellipse;
,,	centroid,	,,	centroid;
,,	in-centre,	,,	point $b+c$, $c+a$, $a+b$.

12. The equations to PP'' and $P'P''$ respectively are

$$\left.\begin{aligned} &\mu\nu\alpha(b\mu-c\nu)+\dots+\dots = 0\\ \text{and}\quad &a\alpha v w(b\mu-c\nu)+\dots+\dots = 0\end{aligned}\right\},\text{ .... (xv.).}$$

13. If g, g' are the centroids of DEF, $D'E'F'$, their coordinates are determined by

$$\left.\begin{array}{lccc} & \lambda\,(b\mu+u), & \mu\,(c\nu+v), & \nu\,(a\lambda+w) \\ \text{and} & \lambda\,(c\nu+u), & \mu\,(a\lambda+v), & \nu\,(b\mu+w) \end{array}\right| \text{, (modulus } 3\Sigma_2\text{)...(xvi.);}$$

hence P'' is the mid-point of gg'; it is also the centroid of the hexagon, and is further the centre of the in-ellipse, touching at the points where AP, BP, CP meet the sides.

14. The conic about the hexagon has for its equation

$$\Sigma\left(\frac{a^2}{\lambda^2 u}\right) = \frac{1}{abc\lambda\mu\nu}\,\Sigma\left\{1+\frac{a^2\lambda^2}{vw}\right\} bc\beta\gamma \text{............(xvii.).}$$

This is a circle when P is the orthocentre, and P' therefore the symmedian point; in fact, the in-triangles are then the "cosine"-triangles, and the circle the "cosine"-circle. We hence obtain the equation to this circle under the form ...

$$\Sigma\,(bc\cos A\,.\,a^2) = \Sigma\,(bc+a^2\cos B\cos C)\,\beta\gamma.$$

15. The D-symmedian line of DEF cuts EF in

$$\lambda\mu^2 ub'^2+\lambda^2\mu a'^2, \quad c\mu^2\nu b'^2, \quad \nu\lambda^2 wa'^2;$$

hence its equation is

$$\nu a\left[ca\mu^2\nu b'^2-\lambda vwa'^2\right]-a\nu\lambda\beta\,(\mu ub'^2+b\lambda^2a'^2)+\mu\gamma v\,(\mu ub'^2+b\lambda^2a'^2)=0;$$

and hence to that through E is

$$\nu aw\,(\nu vc'^2+c\mu^2b'^2)+\lambda\beta\,(ab\nu^2\lambda c'^2-\mu wub'^2)-b\lambda\mu\gamma\,(\nu vc'^2+c\mu^2b'^2)=0.$$

From these two equations we get the symmedian point of DEF to be given by

$$\left.\begin{array}{l} \dfrac{\nu a}{\mu ub'^2+b\lambda^2a'^2}=\dfrac{\lambda\beta}{\nu vc'^2+c\mu^2b'^2}=\dfrac{\mu\gamma}{\lambda wa'^2+a\nu^2c'^2} \\ \text{and of } D'E'F' \text{ by} \\ \dfrac{\mu a}{\nu uc'^2+c\lambda^2a'^2}=\ldots=\ldots \end{array}\right\} \text{.........(xviii.).}$$

16. To find the orthocentre of DEF, we note that the perpendicular from D on EF is also perpendicular to CP; therefore its equation is

$$(\nu\lambda w-\mu^2\cos B v+\lambda^2\mu a\cos A)\,a+(\mu+\lambda\cos C)\,a\nu\lambda\beta$$
$$-(\mu+\lambda\cos C)\,\mu v\gamma=0;$$

and to the perpendicular from E on FD is

$$-(\nu+\mu\cos A)\,\nu w\alpha+(\lambda\mu u-\nu^2\cos Cw+\nu b\mu^2\cos B)\,\beta$$
$$+(\nu+\mu\cos A)\,b\lambda\mu\gamma=0,$$

whence, after reductions, the coordinates of the orthocentre (H_1) are

$$\left.\begin{array}{l}\alpha\equiv(\mu+\lambda\cos C)\left[\lambda\mu+\nu\lambda\cos A+\mu\nu\cos B-\nu^2\cos C\right]\\ \beta\equiv(\nu+\mu\cos A)\left[\mu\nu-\lambda^2\cos A+\lambda\mu\cos B+\nu\lambda\cos C\right]\\ \gamma\equiv(\lambda+\nu\cos B)\left[\nu\lambda+\lambda\mu\cos A-\mu^2\cos B+\mu\nu\cos C\right]\end{array}\right|,$$

$$(\text{modulus } 4R^2/uvw)\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(xix.)}.$$

In like manner, the coordinates of orthocentre (H_2) of $D'E'F'$ are

$$\left.\begin{array}{l}\alpha\equiv(\nu+\lambda\cos B)\left[\nu\lambda+\lambda\mu\cos A+\mu\nu\cos C-\mu^2\cos B\right]\\ \beta\equiv\ldots\\ \gamma\equiv\ldots\end{array}\right\}\ldots\text{(xx.)},$$

with the same modulus as before. From these values we find P'' to be the mid-point of H_1H_2.

17. The perpendicular from P'' on EF (multiplied by EF)

$$=abc\lambda\mu\nu\Delta w/\Sigma_2^2,$$

whence we obtain (xiii.) in another way.

18. If p_1, p_2, p_3; p_1', p_2', p_3' are the perpendiculars from P on EF, FD, DE, and $E'F'$, $F'D'$, $D'E'$ respectively, we have

$$p_1p_2p_3=\frac{b\lambda\mu^2\Sigma_1}{c'\Sigma_2}\times\frac{c\mu\nu^2\Sigma_1}{a'\Sigma_2}\times\frac{a\nu\lambda^2\Sigma_1}{b'\Sigma_2}=p_1'p_2'p_3'.$$

19. The equation to the circle DEF is

$$\Sigma_1.\Sigma_2^2.\Sigma(\alpha\beta\gamma)=\Sigma(a\alpha)\,\Sigma\left\{\mu\nu cw\,(a\lambda b^2\nu-a^2\nu v+cuv)\,\alpha\right\}$$

and to $D'E'F'$ is $\ldots\ \ldots$ ...(xxi.).

$$\Sigma_1.\Sigma_2^2.\Sigma(\alpha\beta\gamma)=\Sigma(a\alpha)\,\Sigma\left\{\mu\nu bv\,(a\lambda c^2\mu-a^2\mu w+buw)\,\alpha\right\}$$

20. If e', f, are the mid-points of $D'F'$, DE respectively, they are given by

$$c\nu\lambda,\ \mu(a\lambda+v),\ \nu w;\quad b\mu\lambda,\ \mu\nu,\ \nu(a\lambda+w);$$

whence Be', Cf intersect in

$$\frac{\alpha}{bc\lambda} = \frac{\beta}{cv} = \frac{\gamma}{bw}, \text{ i.e., on } AP' \text{ ....(xxii.).}$$

21. The circles CDE, AEF, BDF have for their equations

$$\left.\begin{aligned} \Sigma(a\beta\gamma) &= \Sigma(a\alpha)(b\nu w\alpha + a^2\lambda\nu\beta)/\Sigma_2, \quad (a) \\ \Sigma(a\beta\gamma) &= \Sigma(a\alpha)(c\lambda u\beta + b^2\mu\lambda\gamma)/\Sigma_2, \quad (b) \\ \Sigma(a\beta\gamma) &= \Sigma(a\alpha)(a\mu v\gamma + c^2\nu\mu\alpha)/\Sigma_2, \quad (c) \end{aligned}\right\} \text{.........(xxiii.).}$$

The radical axes of (a) and (b), and of (b) and (c) are

$$b\nu w\alpha + \lambda(a^2\nu - cu)\beta - b^2\lambda\mu\gamma = 0,$$
$$-c^2\mu\nu\alpha + c\lambda u\beta + \mu(b^2\lambda - av)\gamma = 0\,;$$

hence the radical centre of the three circles is

$$b\nu\alpha/\left[2ab\lambda\nu\cos A + b\mu v + \nu^2(c^2 - a^2)\right] = \ldots = \ldots.$$

Whence, if P is the orthocentre, the radical centre is the negative Brocard point, and if P is this Brocard point, then the circles pass through the circumcentre.

In like manner, the circle $CD'E'$ has its equation

$$\Sigma(a\beta\gamma) = \Sigma(a\alpha)(b^2\mu\nu\alpha + a\nu w\beta)/\Sigma_2,$$

and the radical centre of the three analogous circles is

$$c\mu\alpha/\left[2ca\lambda\mu\cos A + c\nu w - \mu^2(a^2 - b^2)\right] = \ldots = \ldots.$$

Whence, if P is the orthocentre, then the radical centre is the positive Brocard point, and if P is this point, then the three circles pass through the circumcentre.

22. The radical axis of CDE, $BD'F'$ is

$$\lambda\alpha = \mu\beta\,;$$

therefore the radical axes of these and the analogous circles meet in

$$\alpha\lambda = \beta\mu = \gamma\nu,$$

i.e., in the inverse of P.

23. Since the sides of DEF are parallel to AP, BP, CP, we have

$$\left.\begin{aligned} \cot D &= (\mu\nu - \lambda^2\cos A + \lambda\mu\cos B + \nu\lambda\cos C)/\lambda D \\ \cot E &= (\nu\lambda + \lambda\mu\cos A - \mu^2\cos B + \mu\nu\cos C)/\mu D \\ \cot F &= (\lambda\mu + \nu\lambda\cos A + \mu\nu\cos B - \nu^2\cos C)/\nu D \end{aligned}\right\} \text{...(xxiv.),}$$

where $$D \equiv \lambda \sin A + \mu \sin B + \nu \sin C;$$
hence cot (Brocard angle) $= \Sigma(\lambda^2\mu^2) + \lambda\mu\nu\Sigma(\lambda \cos A)/\lambda\mu\nu D.$

The same result, of course, holds for $D'E'F'$. Compare (xxiv.) with (xix.).

Again, $$DE.DF : AP.BP = 2\Delta ab\lambda\mu\Sigma_1 : \Sigma_2^2;$$
therefore $$DE.EF.FD : AP.BP.CP = 2\Delta abc\lambda\mu\nu\Sigma_1\sqrt{2\Delta\Sigma_1} : \Sigma_2^3.$$

If ρ is the circum-radius of DEF (Δ_1), then, since
$$DE.EF.FD = 4\rho\Delta_1,$$
we have $$AP.BP.CP = \rho\Sigma_2/\Delta, \quad \text{by (xiii.)}.$$

[24. The parallels to AB, BC, CA through D, E, F respectively, are
$$\left.\begin{aligned} a^2\nu\lambda\alpha + ab\nu\lambda\beta - b\mu\nu\gamma &= 0 \\ -c\nu w\alpha + b^2\lambda\mu\beta + bc\lambda\mu\gamma &= 0 \\ ca\mu\nu\alpha - a\lambda u\beta + c^2\mu\nu\gamma &= 0 \end{aligned}\right\} \text{..................(xxv.).}$$

These cointersect in Q, $(b\lambda\mu,\ c\mu\nu,\ a\nu\lambda)$ ....(xxvi.).

In like manner, the parallel through D' to CA is
$$a^2\lambda\mu\alpha - c\nu w\beta + ca\lambda\mu\gamma = 0;$$
and this and the analogous lines for E', F' intersect in Q',
$$(c\nu\lambda,\ a\lambda\mu,\ b\mu\nu) \text{ ....(xxvii.).}$$
QQ' passes through P'', and is bisected by it.

25. The equations to DF', ED' are
$$\left.\begin{aligned} a\mu v\alpha - ca\nu\lambda\beta + c\mu v\gamma &= 0 \\ a\nu w\alpha + b\nu w\beta - ab\lambda\mu\gamma &= 0 \end{aligned}\right\} \text{.............. (xxviii.),}$$
which intersect in $\beta/\mu v = \gamma/\nu w$, *i.e.*, on AP''.

26. We collect a few results of interest.

DE, $D'E'$ intersect on CP'; DF, $E'F'$ on CP;

CF, BE' „ AG; CF', BE on AP_1 (§ 9);

DE, $D'F'$ „ AP.

CF'', FP intersect DE in W so that

$$DW : WE = EE' : EC,$$

and CP' meets AB in L', so that

$$AL' : BL' = u : v.$$

$$\begin{array}{llll}
\text{Again,} & FP'' \text{ cuts } DE \text{ in } R, & DR\ : RE\ = u & : w; \\
 & CP \text{ cuts } FD \text{ in } R', & DR' : R'F = v & : b\mu; \\
 & BP' \text{ cuts } FD \text{ in } R'', & DR'' : R''F = c\nu & : w; \\
 & BP \text{ cuts } DF \text{ in } R_1, & DR_1 : R_1F = a\lambda & : u; \\
\text{and} & BE \text{ cuts } DF \text{ in } R_2, & DR_2 : R_2F = a\lambda b\mu & : wu.
\end{array}$$

27. If $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$, $(\alpha_4, \beta_4, \gamma_4)$ are (for the moment) the coordinates of P, P', P'', G, then

$$\frac{\alpha_1\alpha_2}{\alpha_3\alpha_4} = \frac{\beta_1\beta_2}{\beta_3\beta_4} = \frac{\gamma_1\gamma_2}{\gamma_3\gamma_4};$$

hence, if (1, 2) are inverse points, so also are (3, 4).

28. If P_2 is the isotomic conjugate of P, *i.e.* $(\mu\nu/a^2, \nu\lambda/b^2, \lambda\mu/c^2)$, then P_2P is parallel to $P'P''$, and $P''G = \frac{1}{3}P''P_2$.

29. $$EQ : FQ : DQ = \mu\nu : \nu\lambda : \lambda\mu = F'Q' : D'Q' : E'Q';$$

hence, if P is the in-centre of ABC, Q, Q' are the circumcentres of DEF, $D'E'F'$ respectively.

30. Lines through A, B, C parallel to BP, CP, AP are given by

$$\left.\begin{aligned} b\nu\beta + v\gamma &= 0 \\ c\lambda\gamma + w\alpha &= 0 \\ a\mu\alpha + u\beta &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{xxix.});$$

these intersect in a', b', c', given by

$$(bc\lambda\nu,\ vw,\ -b\nu w),\quad (-c\lambda u,\ ca\mu\lambda,\ wu),\quad (uv,\ -a\mu v,\ ab\mu\nu),$$

and P' is the centre of perspective of ABC, $a'b'c'$. Similarly the parallel through A to CP is given by

$$w\beta + c\mu\gamma = 0,$$

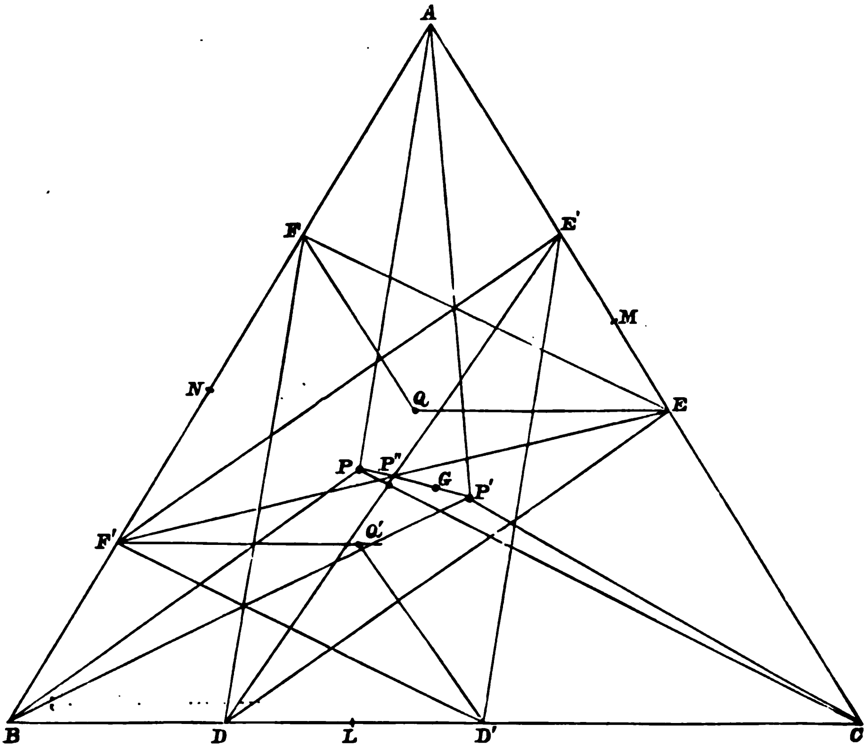
and the three corresponding lines intersect in a'', b'', c'', *i.e.*

$$(wu,\ ca\mu\nu,\ -a\nu w),\quad (-b\lambda u,\ uv,\ ab\nu\lambda),\quad (bc\lambda\mu,\ -c\mu v,\ vw).$$

31. The circle ADD' has for its equation

$$\Sigma(\alpha\beta\gamma) = \Sigma(a\alpha)\, a^2\lambda\mu\nu\left(\frac{c\nu w}{\mu}\beta + \frac{b\mu v}{\nu}\gamma\right) \Big/ \Sigma_1^2 .]$$

32. We now proceed to examine a few cases of envelopes of some of the prominent lines in the figure for different loci of P.



Let P lie on the line $\quad p\lambda + q\mu + r\nu = 0$ ....(A).

From (i.), the equation to DE is

$$\nu v w\alpha + ab\lambda^2\nu\beta - b\lambda\mu v\gamma = 0.$$

Eliminating μ between this equation and (A), we have

$$\nu^3\left[-bcr\alpha\right] + \nu^2\lambda\left[(caq - bcp - abr)\,\alpha + bcr\gamma\right]$$
$$+ \nu\lambda^2\left[(a^2q - abp)\,\alpha + qab\beta + (bcp + abr)\,\gamma\right] + \lambda^3(abp\gamma) = 0.$$

Comparing this with the cubic in Salmon's *Higher Plane Curves* (1873, p. 66), and accentuating the letters in the text-book, we have

$$a' = -bcr\alpha,$$
$$3b' = (caq-bcp-abr)\,\alpha+bcr\gamma,$$
$$3c' = (a^2q-abp)\,\alpha+qab\beta+(bcp+abr)\,\gamma,$$
$$d' = abp\gamma.$$

Then the equation of the envelope of DE is found by putting the above values in

$$a'^2d'^2+4a'c'^3+4b'^3d'-6a'b'c'd'-3b'^2c'^2 = 0.$$

This shows that the envelope is in general a quartic curve.

33. If the locus of P is a straight line passing through an angular point, we have three cases to consider.

(1) $r=0$: the envelope reduces to

$$4b'd' = 3c'^2,$$

i.e. $$4abcp\,(aq-bp)\,\alpha\gamma = \left[a\,(aq-bp)\,\alpha+qab\beta+bcp\gamma\right]^2.$$

This is a parabola touching AB and BC, and having

$$a\,(aq-bp)\,\alpha+abq\beta+bcp\gamma = 0$$

for the chord of contact.

(2) $p=0$: the envelope is $4a'c' = 3b'^2$,

i.e. $$4\,(-abcr\alpha)(aq\alpha+bq\beta+br\gamma) = \left[(caq-abr)\,\alpha+bcr\gamma\right]^2,$$

i.e. the parabola

$$\left[a\alpha\,(cq+br)+bcr\gamma\right]^2+4ab^2cqr\alpha\beta = 0.$$

(3) $q=0$: in this case

$$a' = -bcr\alpha,$$
$$3b' = -b\,(cp+ar)\,\alpha+bcr\gamma,$$
$$3c' = -abp\alpha+b\,(cp+ar)\,\gamma,$$
$$d' = abp\gamma.$$

34. Let P lie on the conic

$$p\mu\nu+q\nu\lambda+r\lambda\mu = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B).$$

Taking the same line (DE) as before, we have to eliminate μ (say) between (i.) and (B). The result is

$$\nu^2\left[c\,(ap-bq)\,a\right]+\nu\lambda\left[(car+a^2p-abq)\,a+abp\beta+bcq\gamma\right]$$
$$+\lambda^2\left[a^2\nu a+abr\beta+abq\gamma\right]=0.$$

The envelope therefore is

$$\left[aa\,(ap-bq+cr)+abp\beta+bcq\gamma\right]^2=4caa\,(ap-bq)(ara+br\beta+bq\gamma),$$

i.e., if $\lambda'=ap-bq-cr,$

$$(aa\lambda'-abp\beta-bcq\gamma)^2+4aba\beta\,(ap-bq\,.\,ap-cr)=0.$$

Hence $a=0$, $\beta=0$ are tangents to the conic, and

$$ca\lambda'-abp\beta-bcq\gamma=0$$

the chord of contact.

35. If $ap=bq=cr$, and the conic (B) therefore the minimum circum-ellipse, the lines DE, &c. become the line at infinity.

36. If (B) is the circumcircle, then the envelope is the parabola

$$4c^2a^2\cos^2 A a^2+a^4\beta^2+b^2c^2\gamma^2+4abc^2\cos A\gamma a+2a^2bc\beta\gamma$$
$$+4ca\,(bc-a^2\cos A)\,a\beta=0.$$

This equation can be written under the form

$$(2ac\cos A a-a^2\beta+bc\gamma)^2+4abc\beta\,(a\gamma+ca)=0;$$

hence $\beta=0,\quad a\gamma+ca=0$

are tangents, and $2ac\cos A a-a^2\beta+bc\gamma=0$

represents the chord of contact.

[37. The equation to DE' is

$$b\mu^2\nu va+a\lambda^2\nu u\beta-\lambda\mu uv\gamma=0\;\ldots\ldots\ldots\ldots\ldots\ldots\text{(C)}.$$

If the locus of P is $p\lambda+q\mu+r\nu=0,$

eliminating ν between these two equations, we get

$$\mu^4\left[bcq^2a\right]+\mu^3\lambda\left[2bcpqa-c^2q^2\gamma-abqra+bcqr\gamma\right]$$
$$+\mu^2\lambda^2\left[bcp^2a+caq^2\beta-2pqc^2\gamma-r\left\{paba+qab\beta-(caq+bcp)\,\gamma\right\}-r^2ab\gamma\right]$$
$$+\mu\lambda^3\left[2capq\beta-c^2p^2\gamma-abrp\beta-capr\gamma\right]+\lambda^4\left[cap^2\beta\right]=0.$$

If we take $p=0$, or $q=0$, this equation reduces to a quadratic, and the envelope can be readily found.

38. The parabola through $DF'AC$ is

$$ca\nu\lambda\beta^2 = c\mu\nu\beta\gamma + b\mu\nu\alpha\beta + 4ca\Sigma_1\gamma\alpha/b^2,$$

the axis being parallel to the median through B.

39. If P moves in the straight line

$$p\alpha + q\beta + r\gamma = 0 \quad \dots\dots\dots\dots\dots\dots\dots\dots (\mathrm{L}),$$

then P' moves in the parallel straight line

$$a\alpha\,(-pbc + qca + rab) + \dots + \dots = 0.$$

Hence, if P moves along a side of the triangle ABC, P' moves along a parallel which bisects the perpendicular from the opposite angle on to that side. And, if P moves on the conic

$$p\beta\gamma + q\gamma\alpha + r\alpha\beta = 0,$$

then P' moves on the conic

$$\Sigma a^2\alpha^2\,(-pa + qb + rc) = 2abc\Sigma\,(p\beta\gamma) \quad \dots\dots\dots\dots (\mathrm{D}).$$

If the primitive locus is the circumcircle, then (D) becomes

$$\Sigma\,(a\alpha^2 \cos A) = \Sigma\,(a\beta\gamma),$$

i.e. the nine-point circle.

If it is the minimum circum-ellipse, then the locus of P' is the maximum in-ellipse.

40. If the locus of P is (L), then the loci of Q, Q' are respectively

$$br\beta\gamma + cp\gamma\alpha + aq\alpha\beta = 0$$

and

$$cq\beta\gamma + ar\gamma\alpha + bp\alpha\beta = 0.$$

The discussion of these and similar results we leave to the reader.]

Thursday, February 9th, 1893.

A. B. KEMPE, Esq., F.R.S., President, in the Chair.

The following papers were read:—

The Harmonics of a Ring: Mr. W. D. Niven.

The Group of Thirty Cubes composed by Six differently coloured Squares: Major MacMahon.

The following presents were made to the Library:—

"Beiblätter zu den Annalen der Physik und Chemie," Band XVI., Stück 12; Band XVII., Stück 1.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. VI., 4th Series.—"James Prescott Joule," by Osborne Reynolds.

"Proceedings of the Royal Society," Vol. LII., No. 317.

"Proceedings of the Royal Irish Academy," Vol. II., No. 3, 3rd Series; December, 1892.

Carruthers, G. T.—"The Cause of Gravity," 8vo; Inverness, 1892.

"Memoirs of the Mathematical Section of the Russian Society of Naturalists," Vol. XIV.; Odessa, 1892.

"The Nautical Almanac for 1896."

"Nyt Tidsskrift for Mathematik," A. 3[die] Aargang, Nos. 7, 8; Copenhagen.

"Nyt Tidsskrift for Mathematik," B. 3[die] Aargang, No. 4; Copenhagen.

"Bulletin of the New York Mathematical Society," Vol. II., No. 4.

"Bulletin des Sciences Mathématiques," 2[me] Série, Tome XVI.; December, 1892.

"Transactions of the Canadian Institute," Vol. III., Part I., No. 5; Toronto, December, 1892.

"Rendiconti del Circolo Matematico di Palermo," Tomo VI., Fasc. 6; November–December, 1892.

"Atti della Reale Accademia dei Lincei — Rendiconti," Vol. I., Fasc. 12, 2° Semestre e Indice; Roma, 1892.

"Atti della Reale Accademia dei Lincei—Memorie," Vol. VI.; Roma, 1890.

"Educational Times," February, 1893.

"Annali di Matematica," Tomo XX., Fasc. 4; Milano, 1893.

"Journal für die reine und angewandte Mathematik," Band CXI., Heft 1; Berlin.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. VI., Fasc. 7–12; Napoli, 1892.

"Indian Engineering," Vol. XII., Nos. 26, 27; Vol. XIII., Nos. 1, 2.

"Washington Naval Observations for 1888," Washington, 1892.

"On Coaxal Systems of Circles," by R. Lachlan, M.A. (extracted from *Quarterly Journal of Pure and Applied Mathematics*, No. 102, 1892). From the author.

On the Thirty Cubes that can be constructed with Six differently Coloured Squares. By Major P. A. MacMahon, R.A., F.R.S.

Read February 9th, 1893. Received March 21st, 1893.

1. It has been long known that the number of rotations which bring a regular solid into coincidence with itself is equal to twice the number of its edges. If, then, a polyhedron possess F faces, and E edges, it is seen that

$$\frac{F!}{2E}$$

different polyhedra may be made by numbering or colouring the faces differently. Thus

	Faces.	Edges.	Number.
Tetrahedron	4	6	2
Cube	6	12	30
Octahedron	8	12	1680
Dodecahedron	12	30	$\frac{12!}{60}$
Icosahedron	20	30	$\frac{20!}{60}$

2. Coming now to the case of the cube, observe that we have found that thirty different cubes may be obtained by numbering the faces with the numbers 1, 2, 3, 4, 5, 6.

Choosing from these a particular cube, say

$$\begin{matrix} & 3 & \\ 2 & 1 & 4 \\ & 5 & \end{matrix}$$

where it is to be understood that 6 is on the face opposite to the face 1 (the face 1 being uppermost), and the remaining numbers are on the remaining (vertical) faces in the circular order shown, observe that this cube remains unaltered for a group of

$$\frac{6!}{30} = 24 \text{ substitutions,}$$

and that *a priori* the order of the group is equal to the number of

rotations of the cube. The group is

1;	(2543);	(24)(35)	(23)(45)(16)	(263)(154)
	(2345);	(24)(16)	(25)(34)(16)	(215)(463)
	(3156);	(35)(16)	(24)(36)(15)	(213)(546)
	(3651);		(24)(13)(56)	(265)(413)
	(2146);		(26)(35)(14)	(256)(143)
	(2641);		(21)(35)(46)	(236)(145)
				(231)(456)
				(251)(436)

and it is singly transitive and imprimitive. It is further holohedrically isomorphous with the group of twenty-four permutations of the four diagonals of the cube.

Exchanging the numbers upon any two opposite faces of the cube we obtain a different cube, which remains unaltered by the same substitutions, and which therefore belongs to the same group as the former cube. These two cubes, whose pairs of opposite faces are marked with the same numbers, which belong to the same group of substitutions, it is convenient to designate "associated cubes."

The thirty cubes are thus separated into fifteen pairs of associated cubes.

Denote the cubes as follows:—

A 3 2 1 4 5	*A'* 5 2 1 4 3	*B* 3 2 1 5 4	*B'* 4 2 1 5 3	*C* 4 2 1 3 5	*C'* 5 2 1 3 4
D 3 1 2 4 5	*D'* 5 1 2 4 3	*E* 3 1 2 5 4	*E'* 4 1 2 5 3	*F* 4 1 2 3 5	*F'* 5 1 2 3 4
G 1 2 3 4 5	*G'* 5 2 3 4 1	*H* 1 2 3 5 4	*H'* 4 2 3 5 1	*I* 4 2 3 1 5	*I'* 5 2 3 1 4
J 3 2 4 1 5	*J'* 5 2 4 1 3	*K* 3 2 4 5 1	*K'* 1 2 4 5 3	*L* 1 2 4 3 5	*L'* 5 2 4 3 1
M 3 2 5 4 1	*M'* 1 2 5 4 3	*N* 3 2 5 1 4	*N'* 4 2 5 1 3	*O* 4 2 5 3 1	*O'* 1 2 5 3 4

where *A*, *A'* are an associated pair, and so on.

The two cubes A, A' have the opposites

$$\begin{array}{c} 1-6 \\ 2-4 \\ 3-5\,; \end{array}$$

rejecting all the cubes which have *any pair* of these opposites, we are left with the following sixteen, viz.:—

$$E,\ F,\ H,\ I,\ K,\ L,\ N,\ O,$$
$$E',\ F',\ H',\ I',\ K',\ L',\ N',\ O'\,;$$

these sixteen may be further divided into two sets, of eight cubes each, which possess a very remarkable and elegant property.

We have: First set $K\ L\ F'\ E'\ H'\ O'\ I\ N$,

Second set $F\ E\ K'\ L'\ I'\ N'\ H\ O$.

In regard to the first set, I say that they are connected with the cube A in the following manner:—

It is possible to form the eight cubes of the set into a single cube in such wise that contiguous faces of the cubes are similarly numbered, and also so that the resulting large cube has four identical numbers exhibited on each face, and from its numbering is identifiable with the cube A.

There are two, and only two solutions, which I exhibit by writing first the lower layer of cubes and beneath it the top layer.

FIRST SOLUTION.

Lower layer.

$$\begin{array}{cc} K & F' \\ L & E' \end{array} \quad \text{or} \quad \begin{array}{cc} \begin{array}{ccc} & 3 & \\ 2 & 4 & 5 \\ & 1 & \end{array} & \begin{array}{ccc} & 3 & \\ 5 & 2 & 4 \\ & 1 & \end{array} \\ \begin{array}{ccc} & 1 & \\ 2 & 4 & 3 \\ & 5 & \end{array} & \begin{array}{ccc} & 1 & \\ 3 & 2 & 4 \\ & 5 & \end{array} \end{array}$$

Upper layer.

$$\begin{array}{cc} H' & I \\ O' & N \end{array} \quad \text{or} \quad \begin{array}{cc} \begin{array}{ccc} & 3 & \\ 2 & 1 & 5 \\ & 6 & \end{array} & \begin{array}{ccc} & 3 & \\ 5 & 1 & 4 \\ & 6 & \end{array} \\ \begin{array}{ccc} & 6 & \\ 2 & 1 & 3 \\ & 5 & \end{array} & \begin{array}{ccc} & 6 & \\ 3 & 1 & 4 \\ & 5 & \end{array} \end{array}$$

SECOND SOLUTION.

Lower Layer.

N O'		3 2 5 1 4	3 1 5 4 2
	or		
I H'		4 2 3 1 5	2 1 3 4 5

Upper layer.

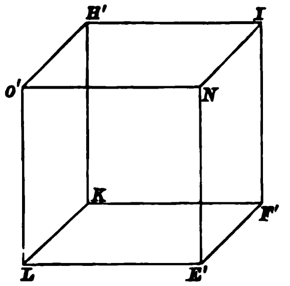
E' L		3 2 1 6 4	3 6 1 4 2
	or		
F' K		4 2 1 6 5	2 6 1 4 5

The second set of eight cubes is similarly connected with the cube A'.

For the examination of this property it is convenient to make a few simple definitions.

I speak of the cube A as *containing* each of the eight cubes

$$K, \quad L, \quad F', \quad E', \quad H', \quad O', \quad I, \quad N.$$



I call the cubes K and N (see figure) *diagonally opposite* with respect to the cube A. So also the pairs L, I; F', O'; E', H' are diagonally opposite with respect to the same cube.

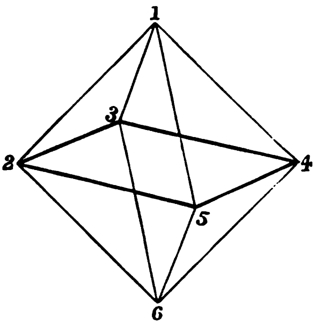
The cubes L, F', H' I speak of as being *adjacent* to K with respect to A; and of the cubes E', I, O' as being *diametrically opposite* to K with respect to A.

It will be evident, as regards the location of the cubes, that in the

example given the second solution is derivable from the first by interchanging the cubes in each diagonally opposite pair.

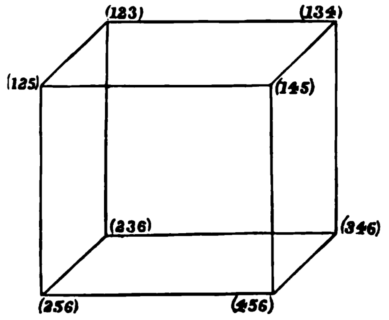
Associated with a cube having numbered faces is an octahedron having numbered summits, formed by joining the middle point of each face with the middle points of the four faces which have with it one edge in common.

Thus the cube *A* yields the octahedron



which may be supposed on a horizontal plane with the diagonal 16 vertical. We have eight octahedral faces having a one-to-one correspondence with the eight summits of the cube. The face 514 corresponds to that summit of the cube which is the point of intersection of the faces numbered 5, 1, 4; the opposite face 236 of the octahedron corresponds to the cube summit determined by the intersection of the faces 2, 3, 6, which is diagonally opposite to the former summit. The latter summit is the cube-summit opposite to the face 514 of the octahedron.

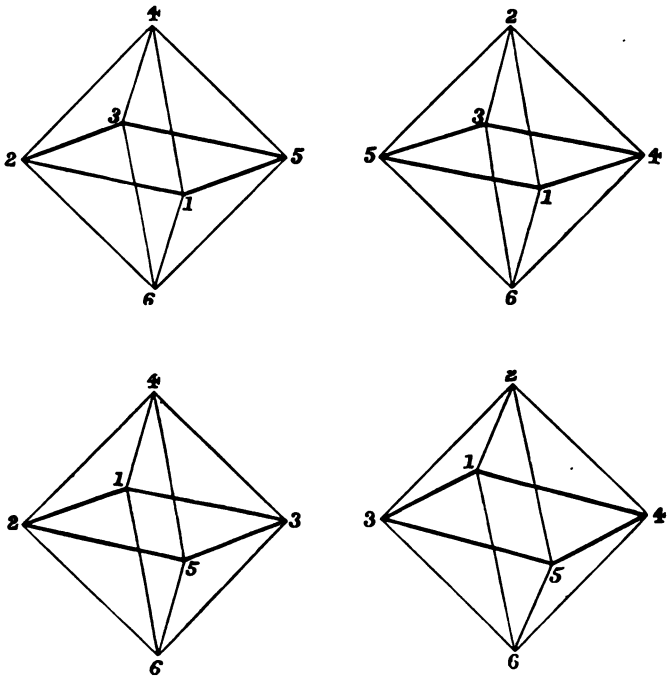
Denote the eight summits of the cube *A* as below.



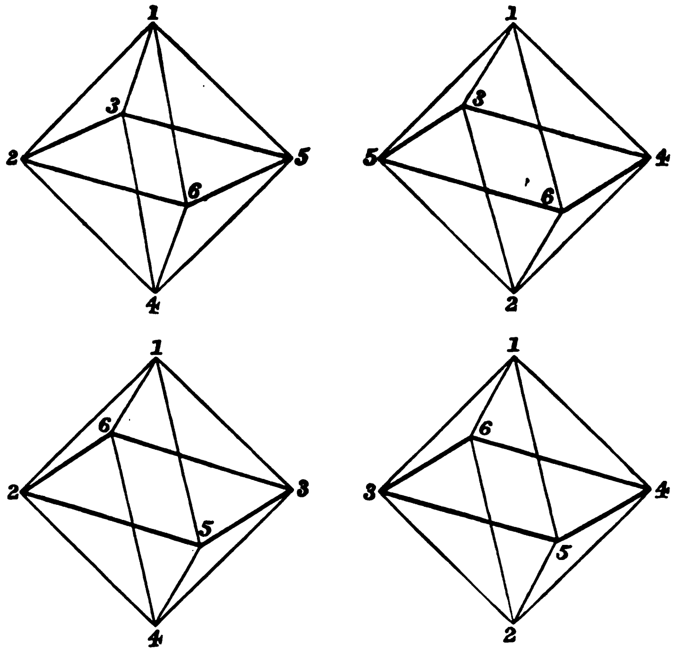
The problem is to properly place the eight octahedra contained by the octahedron A at the eight summits of this cube.

For the summit (236), take the octahedral face 145, which corresponds with the diagonally opposite summit 145, and, regarding the octahedron from an external point, perform the counter-clock-wise substitution (514). The resulting octahedron is to be placed without rotation at the cube summit 236. Similarly, for the summit 346, we perform the clock-wise substitution (512), and place the resulting octahedron without rotation at the summit 346. Proceeding in this way, employing the counter-clock-wise substitution in the cases of those summits which are diametrically opposite to that summit first considered, and clock-wise substitutions for the remaining summits, we obtain the following result:—

Lower layer.



Upper layer.



which constitutes a solution of the problem, and may be identified with the first solution above given. This solution not only selects the proper eight cubes, but places them in their right places and with their right rotations.

The second solution is obtained by merely employing counter-clockwise rotations of octahedral faces where clock-wise rotations are employed in the first solution, and *vice versâ*.

The corresponding substitutions are

Lower layer.	*Upper layer.*
(145), (125),	$(645) \equiv (123)^2$, $(625) \equiv (143)^2$,
(143), (123),	$(643) \equiv (125)^2$, $(623) \equiv (145)^2$.

If a particular cube of the eight be obtained from the containing cube by a circular substitution of the third order, the diagonally opposite cube is obtained by the square of the same substitution.

The condition that a cube Y may be contained by a cube X is clearly that, on replacing them by octahedra, the Y octahedron may

be obtainable from the X octahedron by a circular substitution performed upon the summits which determine a triangular face. In other words, if the Y cube is obtainable from the X cube by a circular substitution performed upon the faces which determine a cube summit, the cube X contains the cube Y.

Hence follows the reciprocal relation between the cubes, and we may say that if a cube X contain the cube Y, then also the cube Y contains the cube X.*

Before proceeding further, it will be convenient to present the whole of the thirty sets of eight cubes.

A	contains	$K L F' E' H' O' I N$,	A'	contains	$F E K' L' I' N' H O$,
B	„	$M O' F D' G' L I' J$,	B'	„	$F' D M' O I J' G L'$,
C	„	$H' G D' E K M' J N'$,	C'	„	$D E' H G' J' N K' M$,
D	„	$L K C' B' G M I' N'$,	D'	„	$C B L' K' I N G' M'$,
E	„	$O' M C A' H K I J'$,	E'	„	$C' A O M' I' J H' K'$,
F	„	$G H' A' B L O J' N$,	F'	„	$A B' G' H J N' L' O'$,
G	„	$C B' O' N F K' D J$,	G'	„	$O N' C' B D' J' F' K$,
H	„	$C' A' L J F' M' E N$,	H'	„	$L' J' C A E' N' F M$,
I	„	$B' A E D' K' M L O$,	I'	„	$E' D B A L' O' K M'$,
J	„	$B C E' F' H O G M$,	J'	„	$E F B' C' G' M' H' O'$,
K	„	$G' I' A C E N D O$,	K'	„	$A' C' G I D' O' E' N'$,
L	„	$N' M' B A I F H D$,	L'	„	$B' A' N M H' D' I' F'$,
M	„	$J L' D E I C' H' B$,	M'	„	$D' E' J' L H B' I' C$,
N	„	$A C' D' F G L' H K$,	N'	„	$D F' A' C H' K' G' L$,
O	„	$A' B' J K G' E' I F$,	O'	„	$J' K' A B I' F' G E$.

In the cube O', J' and E, K' and G, A and F', B and I' are diagonally opposite, respectively, and so on; in every case letters symmetrically placed with regard to the extremities of the row of eight letters denote diagonally opposite cubes.

In any set of cubes, any cube contains the cube diagonally opposite to it, but no other cube of the set. For the cube in question can be seen, by inspection of the substitutions by which the cubes of the set are derived from the containing cube, to be the only cube of the

* Obviously, also, if a cube X contain a cube Y, the cube associated with X contains the cube associated with Y.

set derivable, by a rotation of an octahedral face, from the selected cube.

Suppose cubes X and Y a diagonally opposite pair with respect to Z, and the cube X to be derived by a circular substitution (abc) from Z. Then the substitutions from Z are respectively

X	Y	Z
(abc)	$(abc)^2$	1,

and, performing the substitution (abc), we get

X	Y	Z
$(abc)^2$	1	(abc),

and, again performing the substitution (abc),

X	Y	Z
1	(abc)	$(abc)^2$,

results which establish that Z and X are a diagonally opposite pair with respect to Y, and further that Y and Z are a diagonally opposite pair with respect to X.

Ex. gr., from the above table, we see that

J' and E are diagonally opposite with regard to			O',
O' and J'	,,	,,	E,
E and O'	,,	,,	J'.

This law of reciprocity includes, of course, that previously established.

If, in any set of eight cubes, the cubes W, X, Y be diametrically opposite to Z, it can be shown that the cube which is associated with Z, viz., Z', contains the three cubes W, X, Y. This is obvious on examination of any set of eight cubes as represented by octahedra. Transforming Z to Z', it is found that Z can be transformed into W, X, or Y, by a circular substitution of the third order performed upon some three summits which determine a face of the octahedron.

Each set of eight cubes may be separated into two tetrads of cubes, the cubes in each tetrad being diametrically opposite.

The property of a tetrad is that the cube associated with any cube of the tetrad contains the three other cubes of the tetrad.

Altogether there are sixty tetrads.

Any tetrad of cubes, together with the cube which contains them, further constitute a pentad of cubes, which it is interesting to examine.

The cube A contains the tetrad K, E', O', I.

The pentad is therefore

$$A, \quad K, \quad E', \quad O', \quad I.$$

It can be shown that, from the five cubes

$$A, \quad K, \quad E, \quad O, \quad I,$$

and their associates $\quad A', \quad K', \quad E', \quad O', \quad I'$,

there can be formed altogether ten pentads. Since

$$A \text{ contains } K, E', O', I,$$

$$A, K' \text{ both contain } E', O', I,$$

by the previous proposition.

Therefore $\qquad K'$ contains A', E', O', I,

giving the pentad K', A', E', O', I; and so on.

The ten pentads are

$$\begin{array}{rl}
A; & K, E', O', I, \\
K: & A, E, O, I', \\
E'; & A, K', O, I', \\
O'; & A, K', E, I', \\
I; & A, K', E, O, \\
A'; & K', E, O, I', \\
K'; & A', E', O', I, \\
E; & A', K, O', I, \\
O; & A', K, E', I, \\
I'; & A', K, E', O.
\end{array}$$

The pentad $\qquad A;\ K, E', O', I$,

shows that the cubes

$$A, K' \text{ each contain the three cubes } E', O', I,$$

and that the cubes

$$E', O', I \text{ each contain the two cubes } A, K';$$

and, from the above ten pentads, we find that there are twenty pairs of cubes which contain three cubes in common, and twenty triads of cubes which contain two cubes in common.

There are fifty other pentads, viz., ten each derived from the pentads

$$\begin{array}{l} A;\ L,\ F',\ H',\ N, \\ B;\ M,\ D',\ L,\ I', \\ B;\ O',\ F,\ G',\ J, \\ C;\ H',\ E,\ M',\ J, \\ C;\ G,\ D',\ K,\ N'. \end{array}$$

Altogether there are 120 pairs of cubes which contain three cubes common to each pair, and 120 triads of cubes, the cubes of each triad containing two cubes in common.

Thursday, March 9th, 1893.

A. B. BASSET, Esq., F.R.S., Vice-President, in the Chair.

The following gentlemen were elected members:—F. W. Dyson, M.A., Fellow of Trinity College, Cambridge; J. P. Johnston, M.A. Dub., B.A. Cambridge; T. R. Lee, B.A., late Scholar of Pembroke College, Cambridge; and J. E. A. Steggall, M.A., Professor of Mathematics in University College, Dundee.

Mr. T. J. Dewar exhibited twenty stereographs of the Regular Solids, which were examined with the aid of a stereoscope. He was shown the diagrams furnished by the late Prof. Clerk-Maxwell to the second volume of the *Proceedings.*

The following communications were made:—

Note on the Stability of a Thin Rod loaded vertically: Mr. Love.

On Complex Primes formed with the Fifth Roots of Unity: Prof. Tanner.

On a Threefold Symmetry in the Elements of Heine's Series: Prof. L. J. Rogers.

The Dioptrics of Gratings: Dr. J. Larmor.

The following presents were received:—

A Cabinet Likeness of Mr. Rhodes, presented by Mr. Rhodes.

"Beiblätter zu den Annalen der Physik und Chemie," Band XVII., Stück 2.

"Journal of the Institute of Actuaries," Vol. xxx., Pt. 4, No. 168.

"Proceedings of the Royal Society," Vol. LII., No. 318.

"Mathematical Questions and Solutions," edited by W. J. C. Miller, Vol. LVIII.

"Mittheilungen der Mathematischen Gesellschaft in Hamburg," Band III., Heft 3; 1893.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXVI., Livraisons 4, 5.

"Report of the Superintendent of the U. S. Naval Observatory" for the year ending 1892, June 30.

"Jahrbuch über die Fortschritte der Mathematik," Band XXII., Jahrgang 1890, Heft 1.

"Nieuw Archief voor Wiskunde," Deel xx., Stük 1; Amsterdam, 1893.

"Bulletin de la Société Mathématique de France," Tome xx., Nos. 7 and 8.

"Bulletin of the New York Mathematical Society," Vol. II., No. 5.

"Levensbericht van F. J. van den Berg en Lijst Zijner Geschriften," door D. B. de Haan; Amsterdam, 1893.

"Bulletin des Sciences Mathématiques," Tome XVII.; January, 1893.

"Atti della Reale Accademia dei Lincei—Rendiconti," 5ª Serie, Vol. II., Fasc. 1, 2, Sem. 1; Roma, 1893.

"Annales de la Faculté des Sciences de Toulouse," Tome VI., Pt. 4; 1892.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. VI., Fasc. 1; Napoli, 1893.

"Educational Times," March, 1893.

"Journal für die reine und angewandte Mathematik," Bd. CXI., Heft 2.

"Transactions of the Royal Irish Academy," Vol. xxx., Parts 3 and 4.

"Indian Engineering," Vol. XIII., Nos. 3, 4, 5, and Index to Vol. XII., Pt. 2.

Note on the Stability of a Thin Elastic Rod. *By* A. E. H. Love.

Read March 9th, 1893.

1. The stability of a thin rod or column, vertical when unstrained, and loaded at its upper end, was first investigated by Euler in 1757. He showed that when the load exceeds a certain limit, the rod will be bent under its own weight, and he found the limiting load under which the central line of the rod can take up a form differing very little from the straight form, and crossing its initial position a given number of times. When the load is greater than that needed to produce flexure, and less than that needed to make the central line

cross its initial position anywhere except at ends of the rod, the central line will take up a form of finite curvature which must be one of the curves of the *Elastica* family, and the like conclusion will hold good for all loads which are not precisely those under which the rod can remain indefinitely close to its unstrained position, and indefinitely little bent. This was pointed out by Lamarle in 1846, but I am not acquainted with any general investigation of the stability of the elastica, nor in particular of those forms of it which are indefinitely nearly straight, and have a given number of inflexions. In the present note I propose to treat the case where the lower end of the rod is fixed in position, and the tangent to the central line there is vertical, while the upper end is loaded, reserving the other cases of the problem of the stability of the elastica for another occasion.

2. The equation determining the form of the elastic central line of a naturally straight rod, which is bent in a principal plane by forces and couples applied at its ends alone, can be written down at once, by Kirchhoff's theorem of the *Kinetic Analogue*, in the form

$$B\frac{d^2\phi}{ds^2}+R\sin\phi=0,$$

where B is the flexural rigidity for bending in the plane in question, R the force applied at either end, s the length of the arc measured from a fixed point, and ϕ the angle the tangent to the elastic central line at any point makes with the line of action of R.

We have to consider in particular the case where the rod is held bent by force applied at an end without couple. In this case the flexural couple $-B\,d\phi/ds$ can vanish, and we can write down a first integral of the above equation in the form

$$\tfrac{1}{2}B\left(\frac{d\phi}{ds}\right)^2=R(\cos\phi-\cos\alpha)\ \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where α is the angle which the tangent to the elastic central line at the end where R is applied makes with the line of action of R.

Taking now the end at which R is applied as origin, and the line of action of R as axis of x, the form can be expressed completely in terms of elliptic functions of an argument u, and a modulus k, given by

$$u=s\sqrt{(R/B)},\quad k=\sin\tfrac{1}{2}\alpha\ \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

With these specifications, we have

$$\left.\begin{aligned} x &= 2\sqrt{\frac{B}{R}}\left\{E\operatorname{am}(u+K)-E\operatorname{am}K\right\}-\sqrt{\frac{B}{R}}\,u \\ y &= -2k\sqrt{\frac{B}{R}}\operatorname{cn}(u+K) \end{aligned}\right\} \quad \ldots\ldots\ldots(3),$$

where
$$E\operatorname{am}u = \int_0 \operatorname{dn}^2 u\,du,$$

and K is the real quarter period of the elliptic functions.

This is the form given by Hess, *Math. Ann.*, xxv., 1885.

The points in which the curve cuts the line of action of R are inflexions on the curve.

The parts of the curve between any two consecutive inflexions are exactly equal and similar, and successive parts of this kind lie on opposite sides of the line of action of R. These parts will be called *bays*.

The tangent at the middle point of any bay is parallel to the line of action of R.

The length of the arc of any bay is $2K\sqrt{(B/R)}$, and the distance between consecutive inflexions is $2(2E-K)\sqrt{(B/R)}$.* This distance vanishes when $\alpha = 129{\cdot}3^\circ$ approximately. For greater values of α the force appears to act as a tension pulling out the rod, while for less values it acts as a thrust. When α has the value above stated the curve forms a figure of eight, or a part of such a figure, or several such parts lying one over another.

The figures are well known. They may be inspected in Thomson and Tait's *Natural Philosophy*, Part II., p. 148.

3. I consider now the special case where the rod is built-in vertically at one extremity, and a weight R is attached at the other extremity, the weight being sufficiently great to produce flexure. The curve may (if R is great enough) form a single half bay of an elastica, or an odd integral number of half bays; in other words, there may be one or more $(n+1)$ inflexions. Since the length of a half bay is $K\sqrt{(B/R)}$, and since K is never less than $\frac{1}{2}\pi$, the length of the rod must exceed $\frac{1}{2}\pi\sqrt{(B/R)}$. In order that a form with $(n+1)$ inflexions may be possible, the length of the rod must exceed $\frac{1}{2}(2n+1)\pi\sqrt{(B/R)}$. If the rod is of length $\frac{1}{2}\pi\sqrt{(B/R)}$, the form of

* Here, and in what follows, E is written for $E\operatorname{am}K$.

its elastic central line is a curve of sines of indefinitely small amplitude, and the built-in end is at a middle point of one wave between two inflexions, while the loaded end is at the next inflexion. If the rod is of length l, such that

$$\tfrac{3}{2}\pi\sqrt{(B/R)} > l > \tfrac{1}{2}\pi\sqrt{(B/R)},$$

the curve of the elastic central line is a single half bay of one curve of the elastica family. When $l = \frac{3}{2}\pi\sqrt{(B/R)}$, two forms are possible. One is a single half bay of a curve of the elastica family, and the other consists of three quarters of a complete wave of a curve of sines of indefinitely small amplitude, starting at a point midway between two inflexions, and ending at the next inflexion but one. When

$$\tfrac{5}{2}\pi\sqrt{(B/R)} > l > \tfrac{3}{2}\pi\sqrt{(B/R)},$$

two forms are possible, of which one is a single half bay, and the other three half bays of two different curves of the elastica family.

More generally, when

$$l = \tfrac{1}{2}(2n+1)\,\pi\sqrt{(B/R)},$$

$n+1$ forms are possible, of which n are curves of the elastica family, having respectively $1, 2, \ldots n$ inflexions, and forming respectively $1, 3, \ldots 2n-1$ half bays, and the remaining one is a curve of sines of indefinitely small amplitude having $n+1$ inflexions. Also when

$$(2n+3)\,\pi\sqrt{(B/R)} > l > \tfrac{1}{2}(2n+1)\,\pi\sqrt{(B/R)},$$

$n+1$ forms are possible, which respectively consist of $1, 3, \ldots 2n+1$ half bays of different curves of the elastica family.

4. To investigate the stability of the different forms it is necessary to form an expression for the potential energy in any configuration of equilibrium, and to compare the values of this expression in different configurations of equilibrium. The potential energy depends partly on the position of the weight R above or below the point of support, and partly on the curvature of the elastic central line of the rod.

In any configuration of equilibrium let h be the height of the loaded end above the point of support; then we may take the potential energy due to the raised weight to be Rh. Also the potential energy of strain in the bent rod is

$$\tfrac{1}{2}B\int\left(\frac{d\phi}{ds}\right)^2 ds = R\int(\cos\phi-\cos\alpha)\,ds$$
$$= R\,(h-l\cos\alpha),$$

since, in the notation of § 2,

$$dx/ds = \cos\phi,$$

and h is the extreme value of x. Hence the total potential energy of the system in any configuration is

$$R\,(2h - l\cos a) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Since the height of the next inflexion above the point of support is always obtained from the value of x in equations (3), by putting $u = K$, we find that, when there are $2r+1$ half bays,

$$h = (2r+1)\sqrt{\frac{B}{R}}\,(2E-K) \quad\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where E is written for E am K.

5. We shall now show* that, whenever more than one form is possible with the same length and load, the energy in the configuration with a given number of inflexions is greater than that in any configuration with a smaller number of inflexions.

In the form with $r+1$ inflexions, forming $2r+1$ half bays, let k_1 be the modulus, and E_1, K_1, a_1 the quantities that correspond to E, K, a; and in the form with $s+1$ inflexions, forming $2s+1$ half bays, let k_2, E_2, K_2, a_2 be corresponding quantities.

The condition that the length is the same is

$$(2r+1)\,K_1 = (2s+1)\,K_2.$$

The energy in the form with $r+1$ inflexions is less or greater than that in the form with $s+1$ inflexions according as

$$(2r+1)(4E_1 - 2K_1 - K_1\cos a_1) \lesseqgtr (2s+1)(4E_2 - 2K_2 - K_2\cos a_2),$$

i.e., as $(2r+1)(4E_1 - 3K_1 + 2K_1k_1^2) \lesseqgtr (2s+1)(4E_2 - 3K_2 + 2K_2k_2^2)$,

i.e., as

$$(2r+1)\left[4k_1'^2\left(K_1 + k_1\frac{dK_1}{dk_1}\right) + 2K_1k_1^2\right]$$

$$\lesseqgtr (2s+1)\left[4k_2'^2\left(K_2 + k_2\frac{dK_2}{dk_2}\right) + 2K_2k_2^2\right],$$

* In the course of the work we use the equations

$$E = k'^2\left(K + k\frac{dK}{dk}\right) \text{ and } \frac{d}{dk}\left(kk'^2\frac{dK}{dk}\right) = kK,$$

for which see Cayley's *Elliptic Functions*, Ch. III.

i.e., as $$(2r+1)\,k_1'^2\left(K_1+2k_1\frac{dK_1}{dk_1}\right) \lessgtr (2s+1)\,k_2'^2\left(K_2+2k_2\frac{dK_2}{dk_2}\right),$$

i.e., as $$k_1'^2\left(1+\frac{2k_1}{K_1}\frac{dK_1}{dk_1}\right) \lessgtr k_2'^2\left(1+\frac{2k_2}{K_2}\frac{dK_2}{dk_2}\right).$$

Now $$\frac{d}{dk}\left\{k'^2\left(1+\frac{2k}{K}\frac{dK}{dk}\right)\right\} = -\frac{2kk'^2}{K^2}\left(\frac{dK}{dk}\right)^2,$$

which is always negative; so that the energy in the form with $r+1$ inflexions is less or greater than that in the form with $s+1$ inflexions according as

$$K_1 \gtrless K_2,$$

i.e., according as $$r \lessgtr s.$$

This conclusion is very remarkable. It shows that when any number n of different forms are possible, the energy in any form increases with the number of inflexions in which the curve is cut by the line of action of the load, so that the forms with more than one half bay are all unstable. If we regard the different equilibrium configurations as forming a series, beginning with the configuration in a single half bay and passing to the configurations with 3, 5, ... $2n+1$ half bays, then in each configuration in the series the system has less potential energy than in the one next above it, and more than in the one next below it. It will therefore, if disturbed from any configuration but the one with a single half bay, tend ultimately to pass into the latter configuration.

In particular all the forms which are indefinitely nearly straight, which correspond to loads given by the formula

$$R = \tfrac{1}{4}\,(2n+1)^2\,B\frac{\pi^2}{l^2},$$

are unstable except that for which $n=1$, and the rod under such a load tends to take up a form in which the load hangs below the point of support, and the central line meets the line of action of the load at the loaded end only.

A Geometrical Note. *By* R. TUCKER, M.A.

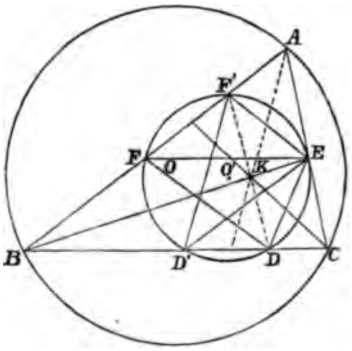
Read December 8th, 1892.

1. If ABC is a triangle, there is one in-triangle which has its sides parallel, and one in-triangle which has its sides antiparallel, to the sides of ABC, and these triangles have a common circumcircle, viz., the nine-point circle of ABC.

Again, there are two in-triangles which have their sides respectively positive and negative oblique isoscelians, and these triangles have a common circumcircle, viz., the sine-triple-angle circle.

2. It is the object of the following note to consider some properties of the six in-triangles, circumscribed in pairs by three circles, which have their sides one a parallel, one an antiparallel, and the remaining one a positive or negative isoscelian.

3. In the figure, EF is parallel to BC, ED antiparallel to AB with respect to C, and DF a positive isoscelian to LB.



The triangle DEF is evidently similar to ABC, and, if its sides are a', b', c', we have

$$\frac{b}{c} = \frac{b'}{c'} = \frac{DE}{EF};$$

but
$$\frac{DE}{CE} = \frac{c}{a} \text{ and } \frac{EF}{AE} = \frac{a}{b},$$

therefore $$\frac{AE}{CE} = \frac{c^2}{a^2},$$

i.e., BE is the symmedian through B.

4. Now, because $$\frac{EF}{a} = \frac{AE}{b} = \frac{c^2}{a^2+c^2},$$

therefore modulus of similarity is

$$ac/(a^2+c^2) = ac/b_1, \text{ say.}$$

Hence
$$\begin{aligned} a' &= a^2c/b_1 = DF, \\ b' &= abc/b_1 = DE, \\ c' &= ac^2/b_1 = EF. \end{aligned}$$

Again,
$$\left.\begin{aligned} BD &= 2a^2c\cos B/b_1 \\ CE &= a^2b/b_1 \\ AF &= c^3/b_1 \end{aligned}\right\} \quad \left.\begin{aligned} CD &= ab^2/b_1 \\ AE &= bc^2/b_1 \\ BF &= ca^2/b_1 \end{aligned}\right\} \ldots\ldots\ldots\ldots(1).$$

5. The trilinear coordinates are for

$$\left.\begin{array}{llll} D, & (0, & b, & 2a\cos B) \\ E, & (a, & 0, & c) \\ F, & (ab, & c^2, & 0) \end{array}\right\} \text{mod.} \equiv b_1 \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

6. Now let us consider the triangle $D'E'F'$, in which $E'D'$ is parallel to AB, $E'F'$ the antiparallel to BC with respect to A, and $D'F'$ a negative isoscelian to LB.

Then we can show, as before, that $D'E'F'$ is similar to ABC, that BE' is the symmedian through B, *i.e.*, E and E' coincide, and that the modulus of similarity is ac/b_1, as before.

Hence $$E D' = a', \quad EF' = b', \quad F'D' = c';$$

and
$$\left.\begin{aligned} BD' &= ac^2/b_1, \\ AF' &= b^2c/b_1, \end{aligned}\right\} \quad \left.\begin{aligned} CD' &= a^3/b_1, \\ BF' &= 2ac^3\cos B/b_1. \end{aligned}\right\}$$

7. Now $$BD.BD' = 2c^3a^3\cos B/b_1^2 = BF.BF',$$

and $$CD.CD' = a^4b^2/b_1^2 = CE^2;$$

therefore the two triangles have a common circumcircle which touches AC at E.

It follows therefore that there are two other circles circumscribed to pairs of triangles, similar to ABC, and which have contact respectively with BC, AB, at the points where they are cut by the symmedians through A, C.

8. The trilinear coordinates are for

$$\left.\begin{array}{ll} D', & (0,\ a^2,\ bc) \\ E', & (a,\ \ 0,\ \ c) \\ F', & (2c\cos B,\ b,\ 0) \end{array}\right\} \text{mod.} \equiv b_1 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

9. From (2), (3), we get the equations to DF', $D'E$, viz.,

$$ab\alpha - 2ca\cos B\beta + bc\gamma = 0,$$

$$ca\alpha + bc\beta - a^2\gamma = 0;$$

these intersect on $$c\beta = b\gamma,$$

i.e., on the symmedian through A, in p_2 say.

In like manner, DF' and EF can be shown to intersect on the symmedian through C, in r_2 say; hence the triangle Ep_2r_2 is in perspective with ABC, and has K for the centre of perspective, and for its symmedian point. It follows, then, that the circles we are considering are Tucker circles (*cf.* Milne's *Companion*, p. 136).

10. The equation to the circle DEF is

$$(a^2+c^2)^2\,\Sigma\,(\alpha\beta\gamma) = (\Sigma a\alpha)\left[bc^3\alpha + 2c^2a^2\cos B\beta + a^3b\gamma\right] \ldots\ldots (4).$$

If O_2 be the centre of this circle, and $\rho_2\ [= Rca/b_1]$ its radius, then the coordinates of O_2 are

$$\rho_2\cos(A-B),\quad \rho_2,\quad \rho_2\cos(B-C);$$

this evidently lies on the line

$$\Sigma\left[bc\,(b^2-c^2)\,\alpha\right] = 0;$$

i.e., on the circum-Brocardal axis, as it should do, by § 9.

In like manner, O_3, O_1, the other centres, are

$$O_3,\quad \cos(C-A),\ \cos(B-C),\ 1;$$

$$O_1,\quad 1,\ \cos(A-B),\ \cos(C-A).$$

11. The coordinates of the "nine-point centre" of ABC are given by

$$\cos(B-C), \quad \cos(C-A), \quad \cos(A-B),$$

and therefore its inverse point by

$$1/\cos(B-C), \quad 1/\cos(C-A), \quad 1/\cos(A-B);$$

hence B_1O_2 and this inverse point are collinear. Similar results hold for O_3, O_1; hence we may see that AO_1, BO_2, CO_3 cointersect in the inverse of the nine-point centre.

12. The equation to the Brocard ellipse is

$$\Sigma(b^2c^2\alpha^2) = 2abc\,\Sigma(\alpha\beta\gamma),$$

and from § 9 we know that the circle DEF has double contact with the ellipse, viz., at E and at b'', given by

$$a(a^2-b^2)^2, \quad b(c^2-a^2)^2, \quad c(b^2-c^2)^2.$$

Similarly c'' is $\quad a(c^2-a^2)^2, \quad b(b^2-c^2)^2, \quad c(a^2-b^2)^2;$

and a'', $\quad a(b^2-c^2)^2, \quad b(a^2-b^2)^2, \quad c(c^2-a^2)^2.$

13. The equation to Eb'', which is, of course, parallel to the join of the Brocard points, because this last is the major axis of the ellipse, is

$$bc\alpha(c^2-a^2)+ca\beta(c^2+a^2-2b^2)-ab\gamma(c^2-a^2) = 0.$$

A slight consideration of the figure shows that the triangle $a''b''c''$ is congruent to the triangle formed by joining the feet of the symmedians of ABC.

Further, Aa'', Bb'', Cc'' cointersect in

$$a/(b^2-c^2)^2, \quad b/(c^2-a^2)^2, \quad c/(a^2-b^2)^2.$$

14. The equations to the tangents at a'', b'', c'', are

$$\frac{bc\alpha}{b^2-c^2}+\frac{ca\beta}{a^2-b^2}+\frac{ab\gamma}{c^2-a^2} = 0 \quad\text{.....................(A)},$$

$$\frac{bc\alpha}{a^2-b^2}+\frac{ca\beta}{c^2-a^2}+\frac{ab\gamma}{b^2-c^2} = 0 \quad\text{.....................(B)},$$

$$\frac{bc\alpha}{c^2-a^2}+\frac{ca\beta}{b^2-c^2}+\frac{ab\gamma}{a^2-b^2} = 0 \quad\text{.....................(C)}.$$

These tangents intersect two and two in

$$a/(c^2-a^2),\quad b/(b^2-c^2),\quad c/(a^2-b^2)\ \text{.........}(\mathrm{A, B});$$

$$a/(b^2-c^2),\quad b/(a^2-b^2),\quad c/(c^2-a^2)\ \text{.........}(\mathrm{B, C});$$

$$a/(a^2-b^2),\quad b/(c^2-a^2),\quad c/(b^2-c^2)\text{............}(\mathrm{C, A}).$$

15. Since the equations to DF, $D'F'$ are

$$2c^2\cos B\alpha-2ab\cos B\beta+b^2\gamma=0,$$

$$b^2\alpha-2bc\cos B\beta+2a^2\cos B\gamma=0,$$

it follows that the join of B to their intersection, and the like joins for the other angles, cointersect in

$$a/(b^2-c^2),\quad b/(c^2-a^2),\quad c/(a^2-b^2).$$

16. The figure shows that $D'F$ is antiparallel to AC with reference to B; hence the circle DEF circumscribes an in-triangle with two sides EF, ED', parallel to BC, AB, and the third side the antiparallel $D'F$.

The Dioptrics of Gratings. *By* J. Larmor, F.R.S.

Read March 9th, 1893.

When a beam of light falls upon a ruled or striated surface, a considerable portion of it is inevitably scattered and lost by the inequalities of the surface; and the residue is reflected or refracted in the ordinary manner. But when the striation varies from point to point in a continuous and fairly uniform way, there is sifted out from the incident beam, in addition to the *débris* of scattered light, a series of regular secondary beams, which are propagated onwards in directions inclined to that of the principal one.

The origin of such a diffracted beam, by the union of the diffracted parts from the different striae which arrive at its front in the same phase, was fully explained by Thomas Young, as also was the very perfect separation of the different chromatic constituents of a regular compound beam by a good grating of this kind. In the few pregnant

sentences in which Young pointed out the reason of these phenomena,* he, in fact, made the way perfectly clear for that extension of their range which was afterwards worked out experimentally by Fraunhofer, and which has more recently led to the development of the optical grating as the chief instrument of spectral analysis."

The discussion of the action of such gratings, so far as it is usually required for practical purposes, is a simple and well-known matter. But there are questions of some importance, such as the effect of want of perpendicularity of the lines of the grating to the plane of incidence of the light, which are more readily attacked by means of a general theory; while it may also be of interest to formally include general diffracted beams within the domain of dioptrical analysis, and exhibit the rules by which the position of their focal lines, when narrow, and the determination of their caustic surfaces in other cases, is to be accomplished.

The fundamental physical principle is that the existence of a continuous wave-front requires either (i) that the optical path measured up to it of the rays which come from all the striae shall be the same, or (ii) that for successive striae it shall differ by the index multiplied by a multiple of a wave-length of the diffracted beam, say by $n\mu_2\lambda_2$ $(=n\mu_1\lambda_1)$ for the diffracted beam of the nth order. Thus, if m be the number of striations between a selected point on the grating and any origin of reference, the difference of paths for the corresponding rays will be $mn\mu\lambda$. This expression will be a function of the coordinates of the point on the grating; and to obtain the Hamiltonian characteristic function of the diffracted beam we have simply to add this function to the characteristic of the unbroken incident beam.

Let us take the equation of the surface of the grating to be, up to the second order,

$$\zeta = \tfrac{1}{2}\alpha\xi^2+\tfrac{1}{2}\beta\eta^2+\nu\xi\eta+\dots;$$

and let the lines of the grating be parallel to the axes of η, so that

$$nm\mu\lambda = L\xi+\tfrac{1}{2}\alpha'\xi^2+\tfrac{1}{2}\beta'\eta^2+\nu'\xi\eta+\dots,$$

where the coefficients α', β', ν' represent the effect of any continuous change in the breadths of the striae that may exist. Suppose the characteristic function of an incident beam in the medium of index μ to be

$$V_1 = \mu_1\left\{l_1\xi_1+m_1\eta_1+n_1\zeta_1+\tfrac{1}{2}A_1\xi_1^2+\tfrac{1}{2}B_1\eta_1^2+\tfrac{1}{2}C_1\zeta_1^2 + F_1\eta_1\zeta_1+G_1\zeta_1\xi_1+H_1\xi_1\eta_1+\dots\right\};$$

* *Phil. Trans.*, 1801.

while the characteristic function of the n'th diffracted beam in the medium of index μ_2 is given by the similar expression with suffix 2, the value of λ_2 above also belonging to this medium; the case of reflexion is obtained by making $\mu_2 = -\mu_1$.

At the surface of the grating we must have

$$V_2 - V_1 = nm\mu\lambda.$$

Hence, considering first the terms of the first degree, we have

$$\mu_2 l_2 - \mu_1 l_1 = L,$$

$$\mu_2 m_2 - \mu_1 m_1 = 0.$$

As these relations are linear, they express that the projection of the incident ray on a normal plane parallel to the lines of striation is bent according to the ordinary law of refraction; while its projection on the normal plane at right angles to these lines is bent in the same manner as an actual ray in this direction would be diffracted, the angles of incidence and diffraction being connected by the relation

$$\mu_2 \sin\phi_2 - \mu_1 \sin\phi_1 = L,$$

where L/n is the value of $\mu\lambda$ divided by the width of a striation at the origin.

The direction of the diffracted beam being thus determined, it remains to find its focal lines. This is done by equating the terms of the second order at the diffracting surface; the equation of the surface must be used to eliminate ζ, and then the two sides of the equation of condition must agree identically. There results

$$(\mu_2 A_2 + n_2 a) - (\mu_1 A_1 + n_1 a) = a',$$

$$(\mu_2 B_2 + n_2 \beta) - (\mu_1 B_1 + n_1 \beta) = \beta',$$

$$(\mu_2 H_2 + n_2 v) - (\mu_1 H_1 + n_1 v) = v';$$

the other coefficients only entering in the third order.

These remaining coefficients are, however, determined by the characteristic equation

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 = \mu^2,$$

which requires

$$(l+A\xi+H\eta+G\zeta)^2+(\ldots)^2+(\ldots)^2 = 1;$$

and this is satisfied up to the first degree of small quantities, provided

$$Al+Hm+Gn = 0,$$

$$Hl+Bm+Fn = 0,$$

$$Gl+Fm+Cn = 0.$$

These equations determine G, F, C in terms of A, B, H.

Now the distances of the focal lines of the beam are the radii of principal curvature, at the origin, of the surface $V = 0$. These radii are equal to the squares of the semi-axes of the central section of the surface

$$\tfrac{1}{2}A\xi^2+\ldots+F\eta\zeta+\ldots = 1$$

by the plane l, m, n; therefore they are determined by making $R = \xi^2+\eta^2+\zeta^2$ a maximum or minimum, subject to the condition

$$l\xi+m\eta+n\zeta = 0.$$

And the analysis may be completed for any special case by means of well-known formulæ in Geometry of Three Dimensions.

A manageable case arises when the incident and diffracted rays are in the same plane, which is therefore normal to the striations. We may now refer each of the beams to its own principal axes. Thus

$$V_1 = \mu_1\,\{z_1+\tfrac{1}{2}A_1x_1^2+\tfrac{1}{2}B_1y_1^2+H_1x_1y_1+\ldots\},$$

$$V_2 = \mu_2\,\{z_2+\tfrac{1}{2}A_2x_2^2+\tfrac{1}{2}B_2y_2^2+H_2x_2y_2+\ldots\};$$

and we will gain symmetry by altering the equation of the diffracting surface to

$$0 = \zeta+\tfrac{1}{2}\alpha\xi^2+\tfrac{1}{2}\beta\eta^2+\nu\xi\eta.$$

Change of coordinates is effected by equations of the type

$$\left.\begin{aligned} x &= \xi\cos\phi-\zeta\sin\phi, \\ z &= \xi\sin\phi+\zeta\cos\phi, \\ y &= \eta. \end{aligned}\right\}$$

On eliminating ζ as before, and so identifying at the surface the two sides of the equation of condition, we have

$$\mu_2 \sin \phi_2 - \mu_1 \sin \phi_1 = L_2,$$

$$\mu_2 A_2 \cos^2 \phi_2 - \mu_1 A_1 \cos^2 \phi_1 = a_2 + (\mu_2 \cos \phi_2 - \mu_1 \cos \phi_1)\, a,$$

$$\mu_2 B_2 \qquad - \mu_1 B_1 \qquad = \beta_2 + (\mu_2 \cos \phi_2 - \mu_1 \cos \phi_1)\, \beta,$$

$$\mu_2 H_2 \cos \phi_2 - \mu_1 H_1 \cos \phi_1 = v_2 + (\mu_2 \cos \phi_2 - \mu_1 \cos \phi_1)\, v.$$

In both the general problem and this more special case, it is to be observed that, if a', β', v' are null, *i.e.*, if the striations are symmetrical with respect to the origin, the focal lines are determined by exactly the same formulæ as would apply to simple refraction at the surface, the different direction of the diffracted ray being allowed for. In the case of Rowland's spherical gratings, this result is well known, and is made use of in the instrumental arrangements. The aberration would be expressed by terms of the third degree.

When the incidence is direct, the circumstances of the diffracted beam will be correctly represented by imagining it to be refracted at an ideal surface situated at each point a distance $mn\mu\lambda / (\mu_2 - \mu_1)$ in front of the real one. But this rule must be modified when the incidence is oblique. The ideal surface would then vary with the angle of incidence, the distance being now $mn\mu\lambda / (\mu_2 \cos \phi_2 - \mu_1 \cos \phi_1)$; for the interposition of a thickness t of medium of index μ_2 retards the ray by an amount that corresponds to a length

$$(\mu_2 \cos \phi_2 - \mu_1 \cos \phi_1)\, t/\mu_1$$

in the medium of index μ_1. The direction of the diffracted ray will be determined by the rule given above; and, once that direction is found, the form of the diffracted beam will be given, when the striation is symmetrical, by the formulæ which belong to ordinary refraction at the surface of the grating.

Another case of some theoretical interest arises when the lines of the grating are closed curves drawn round its centre. If we take

$$nm\mu\lambda = \tfrac{1}{2} a' \xi^2 + \tfrac{1}{2} \beta' \eta^2 + v' \xi\eta + \ldots,$$

that is, if we make L null in the above analysis, these lines will be, in the neighbourhood of the vertex, the system of similar concentric conics $nm\mu\lambda$ = constant, the successive rings enclosed between them being now of equal area. The result indicated by the formulæ is that *the diffracted beam* follows the same direction as the principal

refracted beam, but the equations which give its elements differ by the terms α', β', ν' on their right-hand sides. If the incidence is direct, the grating by itself acts in the same manner as a thin astigmatic lens, whose thickness t is given by

$$(\mu_2-\mu_1)\, t = nm\mu\lambda\,;$$

if it is oblique at an angle ϕ_1, the law of thickness of the equivalent lens is

$$(\mu_2 \cos\phi_2-\mu_1\cos\phi_1)\, t = nm\mu\lambda.$$

On a Three-fold Symmetry in the Elements of Heine's Series. By L. J. Rogers. Received March 8th, 1893. Read March 9th, 1893.

In Heine's *Kugelfunctionen*, Vol. I., Appendix to Chap. 2, it is proved that the series

$$1+\frac{(1-a)(1-b)}{(1-q)(1-c)}\,x+\frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-q)(1-q^2)(1-c)(1-cq)}\,x^2+\ldots$$

$$\equiv \phi\left[a,\, b,\, c,\, q,\, x\right]$$

$$=\prod_{n=0}^{\infty}\frac{(1-axq^n)(1-bq^n)}{(1-xq^n)(1-cq^n)}\,\phi\left[\frac{c}{b},\; x,\; ax,\; q,\; b\right]\;\ldots\ldots(1),$$

which, by the symmetry between a and b, and by reapplication of the same formula, leads to other equivalent forms all consisting of infinite products multiplied by a single series of the form

$$\phi\left[a, b, c, q, x\right];$$

for example, $$\prod_{n=0}^{\infty}\frac{(1-bxq^n)\left(1-\frac{c}{b}q^n\right)}{(1-xq^n)(1-cq^n)}\,\phi\left[b,\, \frac{abx}{c},\, b,\, x,\, q,\, \frac{c}{b}\right]\;\ldots\ldots(2),$$

and $$\prod_{n=0}^{\infty}\frac{\left(1-\frac{abx}{c}q^n\right)}{(1-xq^n)}\,\phi\left[\frac{c}{a},\, \frac{c}{b},\, c,\, q,\, \frac{abx}{c}\right]\;\ldots\ldots\ldots\ldots(3).$$

Suppose $\quad a = \mu e^{-\theta i},\quad b = \nu e^{-\theta i},\quad c = \mu\nu,\quad x = \lambda e^{\theta i},$

so that $e^{-\theta i} = \sqrt{\dfrac{ab}{c}},\quad \lambda = x\sqrt{\dfrac{ab}{c}},\quad \mu = \sqrt{\dfrac{ac}{b}},\quad \nu = \sqrt{\dfrac{bc}{a}};$

then the equations become

$$\phi\left[\mu e^{-\theta i},\ \nu e^{-\theta i},\ \mu\nu,\ q,\ \lambda e^{\theta i}\right]$$

$$= \prod_{n=0}^{\infty} \frac{(1-\lambda\nu q^n)(1-\nu e^{-\theta i} q^n)}{(1-\lambda e^{\theta i} q^n)(1-\mu\nu q^n)}\,\phi\left[\mu e^{\theta i},\ \lambda e^{\theta i},\ \lambda\mu,\ q,\ \nu e^{-\theta i}\right]$$

$$= \prod_{n=0}^{\infty} \frac{(1-\lambda\nu q^n)(1-\mu e^{\theta i} q^n)}{(1-\lambda e^{\theta i} q^n)(1-\mu\nu q^n)}\,\phi\left[\nu e^{-\theta i},\ \lambda e^{-\theta i},\ \lambda\nu,\ q,\ \mu e^{\theta i}\right]$$

$$= \prod_{n=0}^{\infty} \frac{(1-\lambda e^{-\theta i} q^n)}{(1-\lambda e^{\theta i} q^n)}\,\phi\left[\mu e^{\theta i},\ \nu e^{\theta i},\ \mu\nu,\ q,\ \lambda e^{-\theta i}\right].$$

The last of these expressions shows that

$$\phi\left[\mu e^{-\theta i},\ \nu e^{-\theta i},\ \mu\nu,\ q,\ \lambda e^{\theta i}\right]\prod_{n=0}^{\infty}(1-\lambda e^{\theta i} q^n)$$

is a rational function of θ; while the equality of the first and second expressions shows that

$$\phi\left[\mu e^{-\theta i},\ \nu e^{-\theta i},\ \mu\nu,\ q,\ \lambda e^{\theta i}\right]\prod_{n=0}^{\infty}(1-\lambda e^{\theta i} q^n)(1-\mu\nu q^n)$$

$$= \phi\left[\lambda e^{-\theta i},\ \nu e^{-\theta i},\ \lambda\nu,\ q,\ \mu e^{\theta i}\right]\prod_{n=0}^{\infty}(1-\mu e^{\theta i} q^n)(1-\lambda\nu q^n)$$

Now the right side of this equation may be obtained from the left by interchanging λ and μ, while the left side is already known to be symmetrical in μ and ν.

Hence the whole transformation formula for Heinean series is expressed concisely as follows :—

If $\psi(\lambda, \mu, \nu, q, \theta)$ *denote the product*

$$\phi\left[\mu e^{-\theta i},\ \nu e^{-\theta i},\ \mu\nu,\ q,\ \lambda e^{\theta i}\right]\prod_{n=0}^{\infty}(1-\lambda e^{\theta i} q^n)(1-\mu\nu q^n)\ \ldots\ldots\ldots(4),$$

then $\psi(\lambda, \mu, \nu, q, \theta)$ *is a rational function* of θ, *and is symmetrical in* λ, μ, ν.

It remains now to show how this function may be expanded *symmetrically*.

Heine has already proved that

$$\phi\left[a, bq, cq, q, x\right]-\phi\left[a, b, c, q, x\right]=\frac{(1-a)(b-c)}{(1-c)(1-cq)}\phi\left[aq, bq, cq^2, q, x\right].$$

Now the first function can be obtained from the second by writing νq for ν, and the third follows by also writing μq for μ.

This equation therefore gives rise to the following:—

$$\psi(\lambda, \mu, \nu, q, \theta)-(1-\mu\nu)\,\psi(\lambda, \mu q, \nu, q, \theta)$$
$$+\mu\lambda(1-2\nu\cos\theta+\nu^2)\,\psi(\lambda, \mu q, \nu q, q, \theta)=0 \quad\text{......(5).}$$

Let $\qquad\qquad \psi(\lambda, \mu, \nu, q, \theta)$

$$\equiv \chi(\lambda, \mu, \nu, q, \theta)\,\Pi\,(1-2\lambda q^n\cos\theta+\lambda^2 q^{2n})(1-2\mu q^n\cos\theta+\mu^2 q^{2n})$$
$$\times(1-2\nu q^n\cos\theta+\nu^2 q^{2n}),$$

so that (5) becomes

$$(1-2\mu\cos\theta+\mu^2)\,\chi(\lambda, \mu, \nu, q, \theta)-(1-\mu\nu)\,\chi(\lambda, \mu q, \nu, q, \theta)$$
$$+\lambda\mu\chi(\lambda, \mu q, \nu q, q, \theta)=0 \quad\text{......(6).}$$

By the principle of symmetry established above, we see that

$$(1-2\nu\cos\theta+\nu^2)\,\chi(\lambda, \mu, \nu, q, \theta)-(1-\mu\nu)\,\chi(\lambda, \mu, \nu q, q, \theta)$$
$$+\lambda\nu\chi(\lambda, \mu q, \nu q, q, \theta)=0.$$

Eliminating the last function, we have

$$\frac{\chi(\lambda, \mu, \nu, q, \theta)-\chi(\lambda, \mu q, \nu, q, \theta)}{\mu}=\frac{\chi(\lambda, \mu, \nu, q, \theta)-\chi(\lambda, \mu, \nu q, q, \theta)}{\nu},$$

each of which fractions, by symmetry,

$$=\frac{\chi(\lambda, \mu, \nu, q, \theta)-\chi(\lambda q, \mu, \nu, q, \theta)}{\lambda} \quad\text{..................(7).}$$

From these equations we may expand $\chi(\lambda, \mu, \nu, q, \theta)$ in the form

$$A_0+A_1H_1+A_2H_2+\ldots \quad\text{......................(8),}$$

where H_r is a homogeneous symmetrical function of degree r in λ, μ, ν, and where the A's are functions of θ and q only.

Let us, for instance, calculate H_r, or $H_r(\lambda, \mu, \nu)$ say, when it is necessary to specify the elements of the function.

Then, from (7), we evidently have

$$\frac{H_r(\lambda,\mu,\nu)-H_r(\lambda q,\mu q,\nu)}{\lambda}=\frac{H_r(\lambda,\mu,\nu)-H_r(\lambda,\mu q,\nu)}{\mu}=\ldots \qquad \ldots(9).$$

Let the coefficient of $\lambda^\alpha\mu^\beta\nu^\gamma$, where $\alpha+\beta+\gamma=r$, in H_r, be denoted by $a_{\alpha,\beta,\gamma}$.

Then equation (9) shows that, if

$$\alpha+\beta+\gamma=r-1,$$

$$a_{\alpha+1,\beta,\gamma}(1-q^{\alpha+1})=a_{\alpha,\beta+1,\gamma}(1-q^{\beta+1})=a_{\alpha,\beta,\gamma+1}(1-q^{\gamma+1}).$$

Now from this relation we shall be able to evaluate all the coefficients in H_r, assuming, as we obviously may, that

$$a_{r,0,0}=1.$$

Thus
$$a_{r,0,0}(1-q^r)=a_{r-1,1,0}(1-q),$$
$$a_{r-1,1,0}(1-q^{r-1})=a_{r-2,2,0}(1-q^2)=a_{r-2,1,1}(1-q),$$
$$a_{r-2,2,0}(1-q^{r-2})=a_{r-3,3,0}(1-q^3)=a_{r-3,2,1}(1-q).$$

So, too, from $a_{r-3,3,0}$ we can get $a_{r-4,4,0}$ and $a_{r-4,3,1}$, while from the last we can get $a_{r-4,2,2}$.

Theoretically, then, we can completely determine H_r by direct calculation, obtaining a unique solution, so that if by any method we obtain a solution, this solution will be what we seek.

Consider the function

$$\Pi(1-k\lambda q^n)^{-1}(1-k\mu q^n)^{-1}(1-k\nu q^n)^{-1},$$

where Π denotes $\prod\limits_{n=0}^{\infty}$.

Calling this function $P(\lambda,\mu,\nu)$, it is easy to see that

$$\frac{P(\lambda q,\mu,\nu)-P(\lambda,\mu,\nu)}{\lambda}=\frac{P(\lambda,\mu q,\nu)-P(\lambda,\mu,\nu)}{\mu}=\ldots \qquad (10)$$

for all values of k.

Therefore $\Sigma CP(\lambda,\mu,\nu)$ also satisfies (10), where the summation extends to an indefinite number of terms, including arbitrary constants $C_1, C_2, \ldots k_1, k_2, \ldots$.

In other words, a solution of (7) is given by the series

$$A_0+A_1H_1+A_2H_2+\ldots,$$

if H_r denotes the coefficient of k^r in the expansion of $P(\lambda, \mu, \nu)$. Hence this is the unique value of H_r that we were arriving at above.

To determine A_r we may note that, if $\lambda = \mu = 0$, then

$$1 = \Pi\,(1-2\nu q^n \cos\theta + \nu^2 q^{2n})\,\{A_0 + A_1 H_1 + A_2 H_2 + \dots\},$$

where H_r is now the coefficient of k^r in the expansion of

$$\Pi\,(1-k\nu q^n)^{-1},$$

that is $$\nu^r/(1-q)(1-q^2)\dots(1-q^n);$$

therefore

$$A + \frac{A_1\nu}{1-q} + \frac{A_2\nu^2}{(1-q)(1-q^2)} + \dots = \Pi(1-2\nu q^n\cos\theta + \nu^2 q^{2n})^{-1}$$

$$= \Pi\,(1-\nu q^n e^{\theta i})^{-1}\,(1-\nu q e^{-\theta i})^{-1}$$

$$= \left\{1 + \frac{\nu e^{\theta i}}{1-q} + \frac{\nu^2 e^{2\theta i}}{(1-q)(1-q^2)} + \dots\right\}\left\{1 + \frac{\nu e^{-\theta i}}{1-q} + \dots\right\} \dots(11);$$

therefore

$$A_r = 2\cos r\theta + 2\cos(r-2)\,\theta\,\frac{1-q^r}{1-q} + 2\cos(r-4)\theta\,\frac{1-q^r}{1-q}\cdot\frac{1-q^{r-1}}{1-q^2} + \dots$$

$$\dots\dots\dots\dots\dots(12).$$

If r is even, the last term will be independent of θ; but if r is odd, the last term will contain $\cos\theta$.

Collecting the foregoing results, we see that the most general form of Heinean series contains a triple symmetry in its elements, which may be stated as follows:—

$$\phi\left[\mu e^{-\theta i},\ \nu e^{-\theta i},\ \mu\nu,\ q,\ \lambda e^{\theta i}\right]\Pi\,(1-\lambda e^{\theta i} q^n)(1-\mu\nu q^n)$$

$$= \Pi\,(1-2\lambda q^n\cos\theta + \lambda^2 q^{2n})(1-2\mu q^n\cos\theta + \mu^2 q^{2n})(1-2\nu q^n\cos\theta + \nu^2 q^{2n})$$

$$\times\{1 + A_1 H_1 + A_2 H_2 + \dots\}\dots\dots(13),$$

where H_r is the coefficient of k^r in

$$\Pi\,\{(1-k\lambda q^n)(1-k\mu q^n)(1-k\nu q^n)\}^{-1},$$

and $$A_r = (1-q)(1-q^2)\dots(1-q^r)$$

$$\times\{\text{coefficient of } k^r \text{ in } \Pi\,(1-2kq^n\cos\theta + k^2 q^{2n})^{-1}\}.$$

2. Some very interesting results may be derived from this formula by putting $\lambda = 0$. H_r then becomes the coefficient of k^r in

$$\Pi \left\{(1-k\mu q^n)(1-k\nu q^n)\right\}^{-1}.$$

Let $\qquad \mu = xe^{\phi i}$ and $\nu = xe^{-\phi i}$,

and write $A_r(\theta)$ for A_r, considering the latter a function of θ.

Then, evidently, $H_r = \dfrac{A_r(\phi)\,x^r}{(1-q)(1-q^2)\ldots(1-q^r)}$,

so that

$$\frac{\Pi\,(1-x^2q^n)}{\Pi\left\{1-2xq^n\cos(\theta+\phi)+x^2q^{2n}\right\}\left\{1-2xq^n\cos(\theta-\phi)+x^2q^{2n}\right\}}$$
$$= 1+\frac{A_1(\theta)\,A_1(\phi)}{1-q}\,x+\frac{A_2(\theta)\,A_2(\phi)}{(1-q)(1-q^2)}\,x^2+\ldots \qquad \ldots(1).$$

Let a_r denote $A_r(0)$, so that

$$a_1 = 2,$$
$$a_2 = 2+\frac{1-q^2}{1-q} = 3+q,$$
$$a_3 = 2\,(2+q+q^2),$$
$$a_4 = 5+3q+4q^2+3q^3+q^4,$$
$$a_r = 2a_{r-1}-(1-q^{r-1})\,a_{r-2};$$

then, putting $\phi = 0$, we get

$$\frac{\Pi\,(1-x^2q^n)}{\Pi\,(1-2xq^n\cos\theta+x^2q^{2n})^2} = 1+\frac{A_1(\theta)\,a_1}{1-q}\,x+\frac{A_2(\theta)\,a_2}{(1-q)(1-q^2)}\,x^2+\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Again, if $\phi = \frac{\pi}{2}$, we see that $A_r\left(\frac{\pi}{2}\right)$ is the coefficient of

$$k^r/(1-q)\ldots(1-q^r) \text{ in } \Pi\,(1+k^2q^{2n})^{-1},$$

and that $A^{2r}\left(\frac{\pi}{2}\right)$ is therefore

$$(-1)^r\,(1-q)(1-q^3)\ldots(1-q^{2r-1}),$$

while $\qquad A_{2r-1}\left(\frac{\pi}{2}\right) = 0.$

Hence

$$\frac{\Pi\,(1-x^2q^n)}{\Pi\,(1-2x^2q^{2n}\cos 2\theta+x^4q^{4n})}=1-\frac{A_2\,(\theta)}{1-q^2}\,x^2+\frac{A_4\,(\theta)}{(1-q^2)(1-q^4)}\,x^4-\dots$$

$$\dots\dots\dots\dots\dots\dots(3).$$

Again, if $\theta=\phi$, we get

$$\frac{\Pi\,(1-x^2q^n)}{\Pi\,(1-xq^n)^2\,\Pi\,(1-2xq^n\cos 2\theta+x^2q^{2n})}$$

$$=1+\frac{A_1\,(\theta)^2}{1-q}\,x+\frac{A_2\,(\theta)^2}{(1-q)(1-q^2)}\,x^2+\dots\qquad\dots(4),$$

which gives, as a special case,

$$\frac{\Pi\,(1-x^2q^n)}{\Pi\,(1-xq^n)^4}=1+\frac{a_1^2x}{1-q}+\frac{a_2^2x^2}{(1-q)(1-q^2)}+\dots\qquad\dots\dots\dots(5).$$

Again, if $\qquad\nu=\mu q^{\frac12},$

then H_r is the coefficient of k^r in the expansion of

$$\Pi\,(1-k\mu q^n)(1-k\mu q^{\frac12}\cdot q^n),$$

that is, of $\qquad\Pi\,(1-k\mu q^{\frac12 n}),$

which gives $\qquad H_r=\dfrac{\mu^r}{(1-q^{\frac12})(1-q)\dots(1-q^{\frac12 n})}\,.$

Hence

$$\frac{\Pi\,(1-\mu^2q^{n+\frac12})}{\Pi\,(1-2\mu q^{\frac12 n}\cos\theta+\mu^2q^n)}=1+\frac{A_1\,(\theta)}{1-q^{\frac12}}\,\mu+\frac{A_2\,(\theta)}{(1-q^{\frac12})(1-q)}\,\mu^2+\dots$$

$$\dots\dots\dots\dots\dots\dots(6).$$

Comparing (6) with § 1 (11), and remembering that

$$\Pi\,(1-\mu^2q^{2n+1})$$

$$=1-\frac{\mu^2q}{1-q^2}+\frac{\mu^4q^4}{(1-q^2)(1-q^4)}-\frac{\mu^6q^9}{(1-q^2)(1-q^4)(1-q^6)}+\dots,$$

we get a linear relation giving $A_r\,(\theta,\,q^2)$ explicitly and linearly in terms of $A_r\,(\theta)$, $A_{r-1}\,(\theta)\,\dots\,A_1\,(\theta)$, where $A_r\,(\theta,\,q^2)$ denotes the result of writing q^2 for q in $A_r\,(\theta)$.

In the same way, in the general case, if we put

$$\mu=\lambda q^{\frac12},\quad \nu=\lambda q^{\frac32},$$

we get $\quad\dfrac{\phi\,[\lambda q^{\frac12}e^{-\theta i},\,\lambda q^{\frac32}e^{-\theta i},\,\lambda^2,\,q,\,\lambda e^{\theta i}]\,\Pi\,(1-\lambda e^{\theta i}q^n)(1-\lambda^2q^{n+1})}{\Pi\,(1-2\lambda q^{\frac12 n}\cos\theta+\lambda^2q^{n})}$

$$=1+\frac{A_1\,(\theta)}{1-q^{\frac12}}\,\lambda+\frac{A_2\,(\theta)}{(1-q^{\frac12})(1-q)}\,\lambda^2+\dots,$$

while by putting $\mu = \lambda\omega, \quad \nu = \lambda\omega^2,$

where $\omega^2+\omega+1 = 0,$

we get

$$\phi\,[\omega\lambda e^{-\theta i},\ \omega^2\lambda e^{-\theta i},\ \lambda^3,\ q,\ \lambda e^{\theta i}]\ \Pi\,(1-\lambda e^{\theta i}q^n)\,(1-\lambda^3 q^n)$$
$$\div\ \Pi\,(1-2\lambda^3 q^{3n}\cos 3\theta+\lambda^6 q^{6n})$$
$$= 1+\frac{A_3(\theta)}{1-q^3}\lambda^3+\frac{A_6(\theta)}{(1-q^3)(1-q^6)}\lambda^6+\ldots .$$

3. We may also write $\chi(\lambda,\mu,\nu)$ in another form, according to ascending homogeneous functions of μ and ν, which, by the substitution

$$\mu = xe^{\phi i}, \quad \nu = xe^{-\phi i},$$

gives a form for $\chi(\lambda,\mu,\nu)$ in ascending powers of x.

By the definition of $H_r(\lambda,\mu,\nu)$, we see that

$$H_r(\lambda,\mu,\nu) = H_r(\mu,\nu)+\frac{\lambda}{1-q}H_{r-1}(\mu,\nu)$$
$$+\frac{\lambda^2}{(1-q)(1-q^2)}H_{r-2}(\mu,\nu)+\ldots,$$

where $H_r(\mu,\nu)$ stands for $H_r(0,\mu,\nu)$, and is equal to

$$x^r A_r(\phi)/(1-q)(1-q^2)\ldots(1-q^r).$$

$$\chi(\lambda,\mu,\nu) = 1+\frac{A_1(\theta)}{1-q}\{xA_1(\phi)+\lambda\}$$
$$+\frac{A_2(\theta)}{(1-q)(1-q^2)}\left\{x^2A_2(\phi)+x\lambda A_1(\phi)\frac{1-q^2}{1-q}+\lambda^2\right\}$$
$$+\ldots$$
$$= 1+\frac{\lambda}{1-q}A_1(\theta)+\frac{\lambda^2}{(1-q)(1-q^2)}A_2(\theta)+\ldots$$
$$+\frac{xA_1(\phi)}{1-q}\left\{A_1(\theta)+\frac{\lambda}{1-q}A_2(\theta)+\ldots\right\}$$
$$+\frac{x^2A_2(\phi)}{(1-q)(1-q^2)}\left\{A_2(\theta)+\frac{\lambda}{1-q}A_3(\theta)+\ldots\right\}$$
$$+\ldots$$
$$= B_0+\frac{xA_1(\phi)}{1-q}B_1+\ldots,\ \text{say},$$

where the B's are functions of λ and θ only.

Let $\nu = 0$; then the equation becomes

$$\frac{\Pi\,(1-\lambda\mu q^n)}{\Pi\,(1-2\lambda q^n\cos\theta+\lambda^2q^{2n})\,(1-2\mu q^n\cos\theta+\mu^2q^{2n})}$$

$$= B_0+\frac{\mu}{1-q}B_1+\frac{\mu^2}{(1-q)(1-q^2)}B_2+\dots,$$

whence we see that $B_r/(1-q)(1-q^2)\dots(1-q^r)$ is the coefficient of k^r in the expansion of

$$\frac{1}{\Pi\,(1-2\lambda q^n\cos\theta+\lambda^2q^{2n})}\,\Pi\,\frac{1-\lambda kq^n}{1-2kq^n\cos\theta+k^2q^{2n}}.$$

Hence

$$\psi\,(\lambda,\mu,\nu) = \Pi\,(1-2\mu q^n\cos\theta+\mu^2q^{2n})(1-2\nu q^n\cos\theta+\nu^2q^{2n})$$
$$\times\left\{1+B_1H_1(\mu,\nu)+B_2H_2(\mu,\nu)+\dots\right\},$$

where $B_r/(1-q)(1-q^2)\dots(1-q^r)$ is the coefficient of k^r in the expansion of

$$\Pi\,(1-\lambda kq^n)/(1-2kq^n\cos\theta+k^2q^{2n}).$$

Expanding this product, we see that

$$B_r = A_r(\theta)-\lambda A_{r-1}(\theta)\frac{1-q^r}{1-q}+q\lambda^2A_{r-2}(\theta)\,\frac{1-q^r}{1-q}\cdot\frac{1-q^{r-1}}{1-q^2}-\dots.$$

Thursday, April 13th, 1893.

A. B. KEMPE, Esq., F.R.S., President, in the Chair.

Mr. T. S. Barrett was elected a member, and Mr. T. R. Lee was admitted into the Society.

The President mentioned that he had obtained permission from the Council, for reasons which he stated, to alter the title of his recent paper by substituting "Sylvester-Clifford" in the place of "Clifford" only.

The following communications were made:—

Toroidal Functions: Mr. A. B. Basset.

Note on the Centres of Similitude of a Triangle of Constant Form inscribed in a given Triangle: Mr. J. Griffiths.

The Singularity of the Optical Wave-Surface: Mr. J. Larmor.

On a Problem of Conformal Representation: Prof. W. Burnside.

The following presents were received:—

"Beiblätter zu den Annalen der Physik und Chemie," Band XVII., Stück 3; Leipzig, 1893.

"Proceedings of the Royal Society," Vol. LII., No. 319.

"Revue Semestrielle des Publications Mathématiques," Tome 1, 1ère Partie; Amsterdam, 1893.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. VII., No. 1.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XI., No. 3; Coimbra, 1893.

Lemoine (M. Emile)—"La Géométrographie, ou l'Art des Constructions Géométriques," Pamphlet, 8vo; Paris, 1892.

"Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Mathematisch-Physische Classe," IV., V., and VI.; 1892-3.

"Bulletin of the New York Mathematical Society," Vol. II., No. 6.

"Bulletin de la Société Mathématique de France," Tome XXI., Nos. 1 and 2; Paris.

"Bulletin des Sciences Mathématiques," Tome XVII.; Paris, Fév., 1893.

"Journal of the College of Science, Japan," Vol. V., Part 3; Tokyo, 1893.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. II., Fasc. 3, 4, 5, Sem. 1; Roma, 1893.

"Annals of Mathematics," Vol. VII., Nos. 2 and 3; Virginia, 1893.

"Acta Mathematica," XVI., 4; Stockholm, 1893.

"Educational Times," April, 1893.

"Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2, Vol. VI., Fasc. 2; Vol. VII., Fasc. 3; Napoli, 1893.

"Indian Engineering," Vol. XIII., Nos. 7, 8, 9, 10, 11.

"Catalogue of the Michigan Mining School," 1891-2; Houghton, Michigan.

Russell (J. W.)—"Elementary Treatise on Pure Geometry," 8vo; Oxford, 1893.

Lachlan (R.)—"Elementary Treatise on Modern Pure Geometry," 8vo; London, 1893.

Forsyth (A. R.)—"Theory of Functions," Imp. 8vo; Cambridge, 1893.

"American Journal of Mathematics," Vol. XV., No. 1; Baltimore, Johns Hopkins University, 1893.

Works by Prof. G. B. Halsted, Austin, Texas, presented by the author:—

"The Elements of Geometry"; New York, 1885; London, Macmillan, 1893.

"An Elementary Treatise on Mensuration," fourth edition; Boston, 1890.

"Elementary Synthetic Geometry"; New York, 1892.

"The Science Absolute of Space, independent of the Truth or Falsity of Euclid's Axiom XI. (which never can be established *a priori*); followed by the Geometric Quadrature of the Circle in the case of the Falsity of Axiom XI." (translation of paper by J. Bolyai, with three Appendices; extract from the *Scientiæ Baccalaureus*, Vol. I., June, 1891, No. 4).

"The Two-term Prismoidal Formula" (reprint from *Scientiæ Baccalaureus*, Vol. I., February, 1891, No. 3).

"Geometrical Researches on the Theory of Parallels," by N. Lobatschewsky (translated by G. B. H., and the fourth edition, issued for the Council of the Association for the Improvement of Geometrical Teaching, to the members of the Association); Austin, Texas, 1892.

"Number, Discrete and Continuous," preface and first four chapters.

"Note on the First English Euclid" (from the *American Journal of Mathematics*, Vol. II., 1879).

From Dr. A. Macfarlane, Professor of Physics in the University of Texas:—

"The Fundamental Theorems of Analysis Generalized for Space" (read before New York Mathematical Society, May 7th, 1892).

"The Imaginary of Algebra" (a continuation of a paper on the "Principles of the Algebra of Physics"); Salem, Mass., 1892.

Note on the Centres of Similitude of a Triangle of Constant Form inscribed in a Given Triangle. *By* JOHN GRIFFITHS, M.A.

Received April 10th, 1893. Read April 13th, 1893.

I propose in the following note to discuss the following propositions, viz.:—

(1) A triangle DEF inscribed in a given triangle ABC so as to be similar to another given one $A'B'C'$ belongs to some one of twelve systems of similar in-triangles—each system having a centre of similitude of its own.

(2) The centres of similitude of the twelve systems in question can be formed into two groups of six points, which lie, respectively, on two circles, inverse to each other with respect to the circumcircle ABC. If we use isogonal coordinates, the equations of these circles are

$$x \operatorname{cosec} A + y \operatorname{cosec} B + z \operatorname{cosec} C = \cot \omega + \cot \omega',$$

and $$\Sigma x \operatorname{cosec} A = \cot\omega - \cot\omega',$$

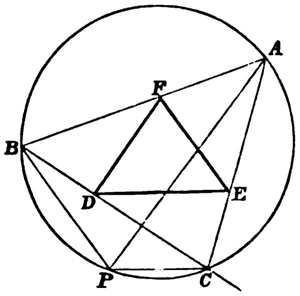
where ω and ω' are the Brocard angles of ABC and $A'B'C'$, respectively.

(3) The centre of similitude of any system of similar triangles inscribed in ABC, and having a common Brocard angle equal to that of $A'B'C'$, will lie on one or other of the above circles.

(4) As a particular case of the problem, I shall also notice the different systems formed by a triangle DEF inscribed in ABC so as to be either directly or inversely similar to it.

SECTION 1.

* When three points D, E, F are taken on the sides BC, CA, AB of a triangle ABC, then the circles CDE, AEF, BFD will intersect in one



or other of the pair of points whose isogonal coordinates are

$$x = \frac{\sin(D+A)}{\sin D}, \quad y = \frac{\sin(E+B)}{\sin E}, \quad z = \frac{\sin(F+C)}{\sin F},$$

and $$x = \frac{\sin(D-A)}{\sin D}, \quad y = \frac{\sin(E-B)}{\sin E}, \quad z = \frac{\sin(F-C)}{\sin F},$$

* If $x, y, z, \alpha, \beta, \gamma$ denote respectively the isogonal and trilinear coordinates of a point, then

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma} = \frac{a\alpha + b\beta + c\gamma}{a\beta\gamma + b\gamma\alpha + c\alpha\beta},$$

where a, b, c are the sides of the triangle of reference ABC. Hence x, y, z satisfy the relation $\Sigma a(x - yz) = 0$, or $\Sigma (x - yz)\sin A = 0$,

it is seen that a circle is represented in this system of coordinates by a linear equation $$\lambda x + \mu y + \nu z = \delta.$$

See a note on "Secondary Tucker-Circles," recently communicated by me to the Society (*Proceedings*, Vol. xxiv., pp. 121 &c.).

where D, E, F denote the angles of the triangle DEF. These two points are inverse to each other with respect to the circumcircle ABC.

It follows from this theorem that a triangle DEF inscribed in ABC so as to be similar to $A'B'C'$, must belong to some one of the six pairs of similar in-triangles whose centres of similitude are the following points, viz.,

$$x = \frac{\sin(D+A)}{\sin D}, \quad y = \frac{\sin(E+B)}{\sin E}, \quad z = \frac{\sin(F+C)}{\sin F},$$

and $$x = \frac{\sin(D-A)}{\sin D}, \quad y = \frac{\sin(E-B)}{\sin E}, \quad z = \frac{\sin(F-C)}{\sin F},$$

where the angles D, E, F have the following values:—

(1) $D = A'$, $E = B'$, $F = C'$; (2) $D = A'$, $E = C'$, $F = B'$;

(3) $D = B'$, $E = C'$, $F = A'$; (4) $D = B'$, $E = A'$, $F = C'$;

(5) $D = C'$, $E = A'$, $F = B'$; (6) $D = C'$, $E = B'$, $F = A'$.

Section 2.

Since the equations $$x = \frac{\sin(D+A)}{\sin D}, \text{ \&c.}$$

give $$\Sigma x \operatorname{cosec} A = \Sigma \cot A + \Sigma \cot D,$$

it is easily seen that the centres of similitude of the six pairs of systems of similar in-triangles considered above must lie on the circles

$$\Sigma x \operatorname{cosec} A = \cot\omega + \cot\omega' \quad \text{and} \quad \Sigma x \operatorname{cosec} A = \cot\omega - \cot\omega',$$

where $\cot\omega = \Sigma \cot A$, and $\cot\omega' = \Sigma \cot D = \Sigma \cot A'$.

It is also evident that any point on either of these two circles can be considered to be the centre of similitude of a certain system of similar triangles inscribed in ABC which have a Brocard angle equal to that of $A'B'C'$.

This may be also stated as follows, viz.: The pedal triangle with respect to ABC of any point P on the above circles has the same Brocard angle as $A'B'C'$. The point P is, in fact, the centre of similitude of triangles inscribed in ABC so as to be directly similar to the pedal of P with respect to ABC.

SECTION 3. *Particular Case of the Problem.*

When the triangle $A'B'C'$ is similar to ABC, the centres of similitude of the six pairs of systems of in-triangles formed by DEF, similar to ABC, are the points

(1) $(2\cos A,\ 2\cos B,\ 2\cos C)$; $(0,\ 0,\ 0)$.

(2) $\left(2\cos A,\ \dfrac{\sin A}{\sin C},\ \dfrac{\sin A}{\sin B}\right)$; $\left(0,\ \dfrac{\sin(C-B)}{\sin C},\ \dfrac{\sin(B-C)}{\sin B}\right)$.

(3) $\left(\dfrac{\sin C}{\sin B},\ \dfrac{\sin A}{\sin C},\ \dfrac{\sin B}{\sin A}\right)$; $\left(\dfrac{\sin(B-A)}{\sin B},\ \dfrac{\sin(C-B)}{\sin C},\ \dfrac{\sin(A-C)}{\sin A}\right)$.

(4) $\left(\dfrac{\sin C}{\sin B},\ \dfrac{\sin C}{\sin A},\ 2\cos C\right)$; $\left(\dfrac{\sin(B-A)}{\sin B},\ \dfrac{\sin(A-B)}{\sin A},\ 0\right)$.

(5) $\left(\dfrac{\sin B}{\sin C},\ \dfrac{\sin C}{\sin A},\ \dfrac{\sin A}{\sin B}\right)$; $\left(\dfrac{\sin(C-A)}{\sin C},\ \dfrac{\sin(A-B)}{\sin A},\ \dfrac{\sin(B-C)}{\sin B}\right)$.

(6) $\left(\dfrac{\sin B}{\sin C},\ 2\cos B,\ \dfrac{\sin B}{\sin A}\right)$; $\left(\dfrac{\sin(C-A)}{\sin C},\ 0,\ \dfrac{\sin(A-C)}{\sin A}\right)$.

Six of these points lie on the Brocard circle

$$\Sigma x \operatorname{cosec} A = 2\cot\omega,$$

and the remaining six on the Lemoine line

$$\Sigma x \operatorname{cosec} A = 0,$$

which is the inverse of the circle in question with respect to the circumcircle ABC.

The two systems whose centres of similitude are

$$\left(\frac{\sin(B-A)}{\sin B},\ \frac{\sin(C-B)}{\sin C},\ \frac{\sin(A-C)}{\sin A}\right)$$

and

$$\left(\frac{\sin(C-A)}{\sin C},\ \frac{\sin(A-B)}{\sin A},\ \frac{\sin(B-C)}{\sin B}\right),$$

i.e., the inverses of the Brocard points $\left(\dfrac{c}{b},\ \dfrac{a}{c},\ \dfrac{b}{a}\right)$, $\left(\dfrac{b}{c},\ \dfrac{c}{a},\ \dfrac{a}{b}\right)$ with respect to the circumcircle ABC, have been already noticed in a paper recently communicated by me to the Society.

In either of these two systems we have a series of in-triangles inversely similar to ABC, and this is also the case with regard to

each of the systems whose centres of similitude are the points

$$\left(2\cos A,\ \frac{a}{c},\ \frac{a}{b}\right);\ \left(\frac{b}{c},\ 2\cos B,\ \frac{b}{a}\right);\ \left(\frac{c}{b},\ \frac{c}{a},\ 2\cos C\right);\ (0,\ 0,\ 0).$$

The point (0, 0, 0) must be regarded as an infinitely distant one on the Lemoine line

$$\Sigma x \operatorname{cosec} A = 0,$$

and, corresponding to each of the remaining six centres of similitude, viz.:—

$$(2\cos A,\ 2\cos B,\ 2\cos C),\quad \left(\frac{b}{c},\ \frac{c}{a},\ \frac{a}{b}\right),$$

$$\left(\frac{c}{b},\ \frac{a}{c},\ \frac{b}{a}\right),\quad \left(0,\ \frac{\sin(C-B)}{\sin C},\ \frac{\sin(B-C)}{\sin B}\right),$$

$$\left(\frac{\sin(C-A)}{\sin C},\ 0,\ \frac{\sin(A-C)}{\sin A}\right),\quad \left(\frac{\sin(B-A)}{\sin B},\ \frac{\sin(A-B)}{\sin A},\ 0\right),$$

the inscribed triangles are directly similar to ABC; that is to say, every one of them may be brought by rotation in its own plane into a position wherein its sides are respectively parallel to the corresponding sides of ABC.

A few theorems with regard to the systems of similar in-triangles whose centres of similitude are the points

$$\left(2\cos A,\ \frac{a}{c},\ \frac{a}{b}\right),\quad \left(\frac{b}{c},\ 2\cos B,\ \frac{b}{a}\right),\quad \left(\frac{c}{b},\ \frac{c}{a},\ 2\cos C\right),$$

i.e., the vertices of the second Brocard triangle of ABC, may be noticed here.

As I have already stated, the in-triangles for each of these points are directly similar to each other and inversely similar to ABC.

If we consider the system of in-triangles corresponding to the centre of similitude $\left(\frac{c}{b},\ \frac{c}{a},\ 2\cos C\right)$, we have the following results, viz.:—

(1) A reference to the figure will show that if $D = B$, $E = A$, and $F = C$, the sides DF, FE, ED are respectively parallel to lines AP, BP, CP which meet in a point P on the circumcircle ABC.

(2) There is no difficulty in finding a geometrical construction for the centre of similitude in question. In fact, if O denote the centre of the circumcircle ABC, the point $\left(\frac{c}{b},\ \frac{c}{a},\ 2\cos C\right)$, or say U, will be the intersection of the circle BAO and that which can be drawn

through B and C to touch AC. If the angle C be acute, the coordinates of U can be written in the forms

$$x = \frac{\sin\theta}{\sin(\theta+A)}, \quad y = \frac{\sin\phi}{\sin(\phi+B)}, \quad z = \frac{\sin\psi}{\sin(\psi+C)},$$

where $\theta = C, \quad \phi = C, \quad \text{and} \quad \psi = \pi - 2C.$

In this case it follows that the angles subtended at U by the sides BC, CA, AB are $\pi - C$, $\pi - C$, and $2C$.

(3) If $D = B$, $E = A$, and $F = C$, the centre of similitude U of the in-triangle DEF (see Fig.) has the following property with regard to DEF, viz., the isogonal point of U with reference to the triangle DEF has the same relation to DEF as U has to ABC.

(4) The centre of similitude of the triangles ABC, DEF, where $D = B$, $E = A$, $F = C$, is a variable point whose locus is the conic represented by the trilinear equation

$$\gamma^2 \sin 2C - \beta\gamma \sin A - \gamma\alpha \sin B + \alpha\beta \sin C = 0.$$

This curve passes through the vertices A, B, and the centroid and orthocentre of ABC. It also touches the symmedian lines AK, BK of the triangle ABC.

(5) If $pa\alpha + qb\beta + rc\gamma = 0$ represent the axis of similitude of ABC, DEF, the envelope of this line seems to be a curve of the third class whose equation is

$$\{a^2p + b^2q + (a^2+b^2-c^2)\,r\}\{qr + pr - 2pq\}$$
$$+ \{(a^2+b^2-c^2)(p+q) + c^2r\}\{pq - r^2\} = 0.$$

(6) If $D = B$, $E = A$, $F = C$, the circumcircle DEF has double contact with the conic

$$\Sigma \sqrt{\sin D \,.\, \sin \overline{D+A}\, x} = 0,$$

or $$\sqrt{\alpha \sin B} + \sqrt{\beta \sin A} + \sqrt{\gamma \sin 2C} = 0.$$

(7) If we take $D = B$, $E = A$, $F = C$, and the other centre of similitude—say, U'—of the in-triangle DEF to be the intersection of the Lemoine line

$$\Sigma x \operatorname{cosec} A = 0$$

with the side AB of the triangle ABC, it will be easily seen that the circle DEF passes through two fixed points, viz., the vertex C and U'. The system whose centre of similitude is the centre of the circumcircle ABC, does not seem to call for special notice. The foci of its double contact inscribed conic are the centre of the circumcircle and orthocentre of ABC.

On a Problem of Conformal Representation.

By Prof. W. Burnside. Read April 13th, 1893.

1. *Introductory.*

The conformal representation of a plane finite polygon, no two of whose sides cross each other, on the half of an infinite plane or on a circle, is due originally to Schwartz. If $a_1\pi$, $a_2\pi$, ... $a_n\pi$ are the internal angles of the polygon in order, and if w and z are complex variables moving respectively in the plane of the polygon and in the infinite half-plane by the functional relation between which the conformal representation is given, it is shown first that

$$\frac{d}{dz}\left(\log\frac{dw}{dz}\right)$$

is everywhere finite, continuous, and uniform in the positive half of the z-plane, and that it takes real values along the axis of real quantities. Hence, by a general theorem, also due to Schwartz, it follows that this function can be continued across the real axis, so that it takes conjugate imaginary values at conjugate points; and therefore that its value at every point of the z-plane is determined by its discontinuities, which all lie on the real axis, and are the points of that axis corresponding to the angular points of the original polygon. If a_1, a_2, ... a_n are these points, the value of the function thus obtained is given by the equation

$$\frac{d}{dz}\left(\log\frac{dw}{dz}\right)=\sum_1^n\frac{a_r-1}{z-a_r}.$$

That the quantities a_1, a_2, &c. are actually determinable in terms of

the lengths of the sides of the polygon has been proved by Herr Schläfli (*Crelle's Journal*, Vol. LXXVIII.).

The formal integral of the above equation is

$$Aw+B=\int (z-a_1)^{\alpha_1-1}\ldots(z-a_n)^{\alpha_n-1}\,dz.$$

The quantity on the right-hand side of this equation is an Abelian integral, so that, although when the region of variation of z is suitably restricted w and z are uniform functions of each other, there will generally be no integral functional relation between w and z for unrestricted variation of the latter.

It may, however, happen in particular cases that the integral is capable of being algebraically transformed into an elliptic integral, and when this is the case, the conformal representation in question will be given by an integral equation of the form

$$f\left[z,\ \wp\,(Aw+B)\right]=0,$$

where f is a symbol for a rational function.

Thus, in the simplest and most familiar case of all,

$$z=\wp w$$

gives the conformal representation of a rectangle on a half-plane, the absolute invariant of the elliptic function being real and greater than unity.

The number of essentially different cases in which this may happen is clearly unlimited, but they are capable of being divided into a limited number of well-defined classes, and it is my object to deal with this question with such generality as the nature of the subject allows.

2. *The General Problem.*

The determination whether in any particular case the integral is capable of being transformed into an elliptic integral, depends, in fact, on geometrical considerations. Suppose that AB and CD are any two sides of the original polygon, and ab and cd the corresponding segments of the real axis in the z-plane.

So long as the variation of z is restricted to the positive half of the z-plane, w is a uniform function of z, and moves within the original polygon. If now z be allowed to pass into the negative half of its plane by crossing the segment ab of the real axis, the rest of the real axis being excluded from its range of variation, w will still be a uni-

form function of z, and it is immediately obvious that the values of w corresponding to the various positions of z in its negative half-plane will be represented by taking the reflection of the original polygon in AB. Similarly, if z be allowed to pass from the positive to the negative half-plane by crossing the segment cd of the axis, the values of w, corresponding to the various positions of z in the negative half-plane, will be represented by the reflection of the original polygon in CD. The process indicated here may be continued indefinitely; and it follows that the various branches of w (each of which is a uniform function so long as the variation of z is restricted to one half-plane) are represented by the successive reflections of the original polygon, and those obtained from it in their sides; those arising from an odd number of reflections corresponding to values of z in the negative half-plane, while an even number of reflections give the branches of w corresponding to values of z in the positive half-plane. Let P now be any point inside the original polygon; then, when the reflecting process is carried out completely, there may be either a finite or an infinite number of different polygons which contain the point P. If the latter is the case, there will correspond an infinite number of values of z to a given value of w, and there can therefore be no relation of the form

$$f(z, \wp w) = 0$$

connecting z and w. But if it is found that there are only a finite number of different polygons containing P, the Abelian integral *must* be capable of transformation into an elliptic integral, as otherwise, when w is given, there would be an infinite number of different values of z.

The problem which must be solved is therefore the following:—To determine the forms of all rectilinear polygons such that any given point of the plane lies within a finite number only of polygons formed by repeated reflections from the original one. It is convenient slightly to alter the form of this question as follows:—A plane polygon and a point P within it are given, and from P is constructed an infinite series of points by successive reflections in any order in the sides of the polygon; it is required to determine whether or no points in the series can be found such that their distances from P, without being zero, can be made less than any assignable quantity; or, in other words, whether the group of transformations arising from reflections in the sides of the polygon is, in M. Poincaré's phrase, a continuous or a discontinuous group. The case of a triangle may

first be dealt with, both as being simplest and as leading to the solution of the more general case.

If the angles of the triangle are not commensurable with two right angles, it is at once clear that the group must be continuous, and that the required reduction of the integral to elliptic form cannot take place.

Suppose, then, that a triangle ABC is placed with the angle A at the origin, and the angle B at the point (1, 0). The effect of a pair of reflections taken successively in AB and AC is given by

$$z' = e^{2iA} z,$$

and that of a pair in BC and AB by

$$z'' - 1 = e^{2iB} (z - 1),$$

where z is $x + iy$ for the original point, and z', z'' the corresponding quantities for the two transformed points. The result of any even number of reflections will be represented by repetitions of these two substitutions and their inverses; and it is sufficient to consider the group of points arising from an even number of reflections, since, by combining these with a single reflection, say in AB, all the others result.

If $\pi - B$ be written for B in the second of the above equations, the only alteration caused is to replace the second substitution by its inverse, so that the group of points is unchanged, while the angles of the triangle are altered from A, B, C to A, $C+A$, $B-A$; and therefore, if one of these triangles leads to a discontinuous group of points, so also does the other.

Now A, B, C are by supposition commensurable, and, if θ is the greatest angle of which they are all integral multiples, it is easy to see that a continued repetition of the process of keeping the smallest angle unchanged and replacing the next least by its supplement, will lead eventually to an isosceles triangle whose equal angles are both θ. It follows that unless this triangle leads to a discontinuous group of points, the original triangle cannot do so; and therefore it is sufficient first to deal with an isosceles triangle. The advantage so gained is that for an isosceles triangle a simple and interesting general expression is readily obtainable for the group of points in question.

Thus, taking now

$$A = B = p\pi/q,$$

where p/q is a proper fraction in its lowest terms, then

$$e^{2iA} = e^{2iB} = \omega,$$

a primitive q^{th} root of unity, and the two substitutions are

$$z' = \omega z, \quad z'' = 1+\omega(z-1).$$

The general expression for any substitutions compounded of these two is easily seen to be

$$\zeta = \sum_1^q m_r \omega^r + \omega^s z, \qquad s = 1, 2, \dots q,$$

where the only relation between the positive and negative integers $m_1, m_2 \dots$ is that their sum is zero.

When $q = 2$, this expression is

$$2m \pm z,$$

where m is any positive or negative integer, and the group is evidently discontinuous. The corresponding triangle has two right angles and infinite area.

When $q = 3$, the expression becomes

$$m+(3n-m)\,\alpha+\alpha^s z, \qquad s = 1, 2, 3,$$

where m, n are any positive or negative integers, and α an imaginary cube root of unity. The group again is obviously discontinuous, and the triangle equilateral.

When $q = 4$, the expression becomes

$$m+(2n-m)\,i+i^s z, \qquad s = 1, 2, 3, 4.$$

The group is again discontinuous and the triangle isosceles and right-angled.

Passing over the case $q = 5$, the expression

$$\sum_1^6 m_r \omega^r + \omega^s z,$$

where ω is a primitive sixth root of unity, is easily thrown into the form

$$m+n\alpha \pm \alpha^s z, \qquad s = 1, 2, 3,$$

where m, n are any positive or negative integers, and α an imaginary cube root of unity.

The group is once again discontinuous, while the triangle has two angles each equal to $\pi/6$.

For any value of q other than 2, 3, 4 or 6, the quantity

$$\sum_1^q m_r \omega^r$$

is expressible as the sum of three or more constant quantities, whose ratios are not real, each multiplied by an arbitrary integer, and, by a

known theorem, can be made less than any assignable quantity without being zero. Hence, for all such values of μ, the series of points will contain an infinite number lying indefinitely near P. It follows, therefore, that the only triangles satisfying the required condition are those whose angles are multiples either of

$$\frac{\pi}{2},\ \frac{\pi}{3},\ \frac{\pi}{4}\ \text{or}\ \frac{\pi}{6}.$$

The only triangles satisfying these conditions are those whose angles are

$$\text{(i.)}\ \frac{\pi}{4},\ \frac{\pi}{4},\ \frac{\pi}{2};\quad \text{(ii.)}\ \frac{\pi}{3},\ \frac{\pi}{3},\ \frac{\pi}{3};$$

$$\text{(iii.)}\ \frac{\pi}{6},\ \frac{\pi}{3},\ \frac{\pi}{2};\quad \text{(iv.)}\ \frac{\pi}{6},\ \frac{\pi}{6},\ \frac{2\pi}{3};$$

and these are therefore the only triangles which can be conformally represented on a half-plane or circle by means of an integral equation between two variables.

It is convenient to add here one four-sided figure, namely a rectangle, for which, from elementary geometrical considerations, the same property evidently holds.

It follows immediately, from the present geometrical point of view, that any simply-connected plane polygon which can be formed by the juxtaposition of any finite number of one of these five figures, all of equal size and with homologous sides contiguous, possesses the same property. For, by repeated reflection in its sides, no new sides and corners can be formed other than those which would arise from the same process applied to its component parts, and hence only a finite number of different polygons can be formed which will contain a given point P. I now propose to go on to show that no plane polygons other than these possess the property in question.

Let AB, BC, CD, DE be four consecutive sides of such a polygon. Then, considering first repeated reflections in AB, BC, and CD, it follows, from what has just been proved, that the angles ABC and BCD must both be multiples either of $\pi/6$ or of $\pi/4$; and the same holds for the angles BCD and CDE, and for each pair of consecutive angles of the polygon. Hence the polygon must be such that its angles are all multiples of $\pi/6$ or all multiples of $\pi/4$. For the sake of clearness of statement I now take a particular case, though it will be at once seen that the nature of the argument is such that it will apply to any case whatever. Suppose that A, B, C, D, E are consecutive angles of a polygon which leads to a discontinuous group

of points, and suppose that the angles ABC, BCD, DCE are respectively $3\pi/4$, $3\pi/4$, and $\pi/2$. Through C draw CF parallel to DE. Then reflections in BC, CD, and BC are equivalent to a single reflection in CF, so that among the operations which are equivalent to an even number of reflections in the sides of the polygon are pairs of reflections in AB, CF, and in AB, DE, respectively.

Now, unless the perpendicular distances between AB and CF and between AB and DE are commensurable with each other, the group of points arising from this repetition and combination of these two operations is continuous. Hence these two distances, or, in other words, $BC\sqrt{2}$ and CD, must be commensurable.

Continuing this process, it may be seen that CD and DE must be commensurable, and generally that the ratio of any two sides of the polygon will be commensurable either with unity or with $\sqrt{2}$; the former being the case when both sides have at both their extremities angles which are odd multiples of $\pi/4$, or when both sides have, at one of their extremities at least, angles which are even multiples of $\pi/4$, the latter occurring when one side has at both extremities odd multiples of $\pi/4$, and the other an even multiple at one extremity at least. But these are obviously the conditions that the polygon should be capable of being divided into a finite number of equal isosceles right-angled triangles. And in a precisely similar way it may be shown that the only other cases that can occur are those in which the polygon can be divided into a finite number of equal rectangles or a finite number of equal triangles with angles $\pi/6$, $\pi/3$, and $\pi/2$, the latter case evidently including those of figures formed from triangles (ii.) and (iv.) of p. 191.

The result thus obtained is a known one as regards the four triangles, but I am not aware that a direct proof has ever been given from the purely geometrical point of view taken in this paper; while, as far as I know, the theorem that every rectilinear polygon, that can be represented conformally on a circle or half-plane by means of an integral relation between two variables, is capable of being formed by the juxtaposition either of equal rectangles or equal triangles of one of the four forms, has not been formally proved before this.

In giving some illustrative examples of the actual equations which lead to representations of the sort here considered, it is convenient to deal separately, on account of the greater symmetry of the formulæ, with those cases which arise from the juxtaposition of equilateral triangles, though strictly they do not form a separate class

from those given by the triangles $\frac{\pi}{6}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$. No corresponding separation need be made in the other case, as two isosceles right-angled triangles, when placed together, form either the same figure again, or a square, which leads merely to a particular case of the class of figures arising from a rectangle.

3. *Method of Solution.*

The actual equation which gives the conformal representation of any such area as is being dealt with, on a half-plane, may be directly obtained by algebraical processes, when once the representation of the simplest constituent area, triangle or rectangle, has been carried out. It is necessary to deal separately with the two cases in which the simple area is either a triangle or a rectangle.

Suppose, then, that a triangle ABC has been represented conformally on a half-plane by an equation

$$f(pw, z) = 0,$$

where w is the variable in the plane of the triangle, and z that in the half-plane, and suppose that to the angular points A, B, C of the triangle correspond the points 0, 1, ∞ of the z-plane—this involving no limitation, since, by a linear transformation performed on z, any three points on the real axis may be brought to coincide with $0, 1, \infty$.

If $A'BC$ be the reflection of ABC in BC, the complete figure $ABA'CA$ is represented by means of the equation

$$f(pw, z) = 0,$$

on a complete z-plane, which is bounded by a slit running along the real axis from 1 through 0 to ∞; or, say upon a positive half-plane and a negative half-plane, which are continuous only along that part of the real axis for which x is positive and greater than unity. If now $A'BC''$ be the reflection of $A'BC$ in $A'B$, the complete figure $ABC''A'CA$ will be represented by means of the same equation on two positive half-planes and one negative half-plane, the two former not being directly connected with each other, while they are continuous with the negative half-plane along the segments $1\,\infty$ and $0\,1$ of the real axis respectively. The area thus formed is a simply-connected plane area bounded by straight lines (namely, segments of the real axis), and has its angles multiples of two right angles —the angles actually being 2π at 0, 3π at 1, and 2π at ∞. It may therefore be represented on a positive half ζ-plane by means of an equation between z and ζ which is algebraical in both variables, and

of the first degree in z. The elimination of z between this equation and $f(pw, z) = 0$ will give the required representation of $ABC'A'CA$ on a half-plane. The process thus indicated for the figure formed by the juxtaposition of three triangles may clearly be carried out in any case, so long as the figure is simply-connected (not excluding cases in which parts of the w-plane are covered more than once). When, however, the number of constituent triangles is sufficiently great, it may happen that some of their angular points do not lie on the boundary of the resultant figure. For instance, in the regular hexagon formed by six equilateral triangles, of the eighteen angular points of the triangles, twelve only lie on the boundary of the hexagon, while the other six coincide at its centre. These six all give the same value of z, and belong to triangles which alternately are represented on positive and negative half-planes; and therefore, in the z-plane figure, the centre of the hexagon corresponds to a branch-point where three complete sheets of the z-plane are cyclically connected.

It may now be shown that the form of the required equation between z and ζ may be written down directly, from inspection of the figure to be represented, when divided into its constituent triangles. Thus, if the figure is formed of N triangles, z is a rational function of ζ of degree N; and since, when ζ is real, z is also real, the coefficients must be real. When ζ is regarded as a function of z, its only branch-points are 0, 1, ∞, since the coefficients of the equation are real, and the half of the corresponding Riemann's surface which is given by the system of half z-planes is only branched at these points; and hence, if the equation be written

$$z : z-1 : 1 :: A\prod_1^r (\zeta-\alpha_1)^{m_1} : B\prod_1^s (\zeta-\beta_1)^{n_1} : C\prod_1^t (\zeta-\gamma_1)^{p_1},$$

it follows that

$$\sum_1^r (m_1-1)+\sum_1^s (n_1-1)+\sum_1^t (p_1-1) = 2N-2,$$

the branch-points being all put in evidence by the equation.

Also, since the three products are all of degree N, it follows that

$$\sum_1^r m_1 = \sum_1^s n_1 = \sum_1^t p_1 = N;$$

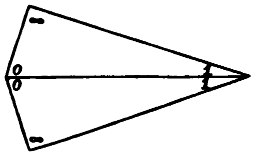
and therefore $$r+s+t = N+2.$$

If any one of the constants α, β, γ is infinity (more than one cannot be so) the corresponding factor must be replaced by a constant.

The values of the indices are given at once by an inspection of the triangle figure. If, at a point on the boundary of the resultant figure,

μ angular points of constituent triangles for which $s = 0$ coincide among the indices m, one will be μ; and if, at any point within the figure, $2\mu'$ such angular points coincide, then μ' will occur twice among the m indices.

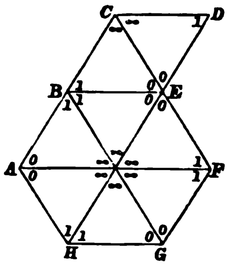
Thus, for the figure



the form of equation is

$$z : z-1 : 1 :: A(\zeta-\alpha_1)^2 : B(\zeta-\beta_1)^2 : C(\zeta-\gamma_1)(\zeta-\gamma_2);$$

while for the figure



the equation is

$$\begin{aligned} z : z-1 : 1 :: & \ A(\zeta-\alpha_1)^2(\zeta-\alpha_2)^2(\zeta-\alpha_3)^4 \\ & : B(\zeta-\beta_1)(\zeta-\beta_2)^2(\zeta-\beta_3)^2(\zeta-\beta_4)^3 \\ & : C(\zeta-\gamma_1)^2(\zeta-\gamma_2)^3(\zeta-\gamma_3)^3. \end{aligned}$$

The number of constants α, β, γ is $r+s+t$, or $N+2$, and the number of relations between them and A, B, C, arising from the identity

$$A\prod_1^r(\zeta-\alpha_1)^{m_1} - B\prod_1^s(\zeta-\beta_1)^{n_1} = C\prod_1^t(\zeta-\gamma_1)^{p_1},$$

is $N+1$. The ratios only of $A : B : C$ are involved, and these are given in terms of the α, β, γ by two of the equations, so that there are $N-1$ equations connecting the $N+2$ quantities α, β, γ.

When three are given the rest are therefore determinate. From the nature of the problem, it follows that those of these constants which correspond to points on the boundary of the figure are necessarily real, while those which correspond to angular points of constituent triangles that lie within the area occur as pairs of conjugate imaginaries. The various determinations of the constants when

three, for points on the boundary, are given, correspond to the various ways in which these points can be chosen.

Thus, for the second of the two figures given, if α_1, β_1, γ_1 have the values 0, 1, ∞ assigned to them, then, while C must correspond to the point ∞ on the real axis in the ζ-plane, either A or G may correspond to the point 0, or either F or H to the point 1.

When the constituent area is a rectangle, the real axis in the z-plane will be divided into four segments, say by the points 0, 1, κ, ∞ across each of which the successive positive and negative half-planes may be continuous. If the resultant figure is formed from N elementary rectangles, z will still be a rational function of ζ of degree N with real coefficients; but now ζ, regarded as a function of z, will have the four branch-points 0, 1, κ, ∞, and the equation between z and ζ will be of the form

$$z : z-1 : z-\kappa : 1$$
$$:: A\prod_1^r(\zeta-\alpha_1)^{m_1} : B\prod_1^s(\zeta-\beta_1)^{n_1} : C\prod_1^t(\zeta-\gamma_1)^{p_1} : D\prod_1^u(\zeta-\delta_1)^{q_1},$$

again putting in evidence all the branch-points.

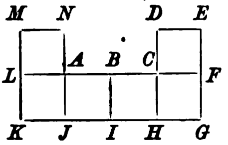
Exactly as before, the indices m, n, p, q are directly determined by an inspection of the figure. The relation between their numbers is now

$$r+s+t+u = 2N+2,$$

and when three of the constants α, β, γ, δ are chosen arbitrarily, the rest are determinate.

In the particular case in which all the reflections of the original rectangle are taken in one pair of opposite sides, the problem becomes the same as that of the transformation of the corresponding elliptic functions.

It may be pointed out here that when the complete area has a line of symmetry, the equations for the determination of the constants may be made more simple by taking account of it. Thus,



if the figure *ABIJKLMN* has been represented on the half ζ-plane, the complete figure formed by this and its reflection in *BI* will be represented on a half ζ'-plane at once by taking ζ a suitable quadratic function of ζ'.

It is, however, obvious, from the known theory of the modular equations, that the equations giving the constants will not generally be algebraically soluble.

4. *Examples.*

For the sake of completeness, I give first the formulæ for the representation of the four triangles. These have already been given in a variety of forms, so that little more than the results, which will be useful for further applications, need here be written down.

Dealing first with elliptic functions for which the absolute invariant J is zero, and taking

$$g_2 = 0, \quad g_3 = -4,$$

the equation

$$4z = \wp'^2 w \quad \text{..............................(i.)}$$

leads to

$$z-1 = \wp^3 w, \quad dz = 3\wp^2 w\, \wp' w\, dw,$$

and

$$\frac{dz}{z^{\frac{1}{2}}(z-1)^{\frac{2}{3}}} = 6\, dw\,;$$

so that equation (i.) gives the representation of a triangle

$$\left(\frac{\pi}{2},\ \frac{\pi}{3},\ \frac{\pi}{6}\right)$$

on the half z-plane, the angles corresponding in the order written to $0, 1, \infty$.

By placing together two such triangles, so as to form an equilateral triangle, equation (i.) represents the equilateral triangle conformally on positive and negative half z-planes which are continuous along the negative part of the real axis; and the equation

$$z : z-1 : 1 :: (2\zeta-1)^2 : 4\zeta(\zeta-1) : 1$$

represents these two half z-planes on a half ζ-plane, so that

$$2(2\zeta-1) = \wp' w \quad \text{.........................(ii.)}$$

represents the equilateral triangle on the half-plane, the angular points of the triangle again corresponding to $0, 1, \infty$.

Thirdly, by placing two of the original triangles together so as to form a triangle $\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3}\right)$, equation (i.) will again represent this triangle on two half z-planes continuous between 0 and 1; while the equation

$$z : z-1 : 1 :: (\zeta+1)^2 : (\zeta-1)^2 : 4\zeta$$

gives the necessary means of passage to a single half-plane. Hence

$$(\zeta+1)^2 = \zeta\wp'^2 w \quad \text{..........................(iii.)}$$

represents the triangle $\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3}\right)$ on the half ζ-plane, the angles corresponding in order to $0, \infty, 1$.

Taking next elliptic functions for which J is unity, and making

$$g_2 = -4, \quad g_3 = 0,$$

the equation
$$1-z = \left(\frac{\wp w - 1}{\wp w + 1}\right)^4 \quad \text{..........................(iv.)}$$

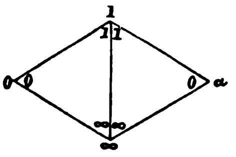
leads to
$$z = \frac{2\wp'^2 w}{(\wp w+1)^4}, \quad dz = -8\,\frac{(\wp w-1)^3 \wp' w\, dw}{(\wp w+1)^5},$$

and
$$\frac{dz}{z^{\frac{1}{2}}(1-z)^{\frac{3}{4}}} = -4\sqrt{2}\, dw;$$

so that equation (iv.) represents the triangle $\left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right)$ on a half-plane, the angles corresponding in order to $1, \infty, 0$.

It will not now be perhaps without interest to illustrate the method of the preceding section by dealing with a certain number of special cases of figures which can be constructed from these triangles. The numbers $0, 1, \infty$, placed inside the figures, indicate the points of the original z-plane to which the angles of the constituent triangle correspond, while the numbers placed outside the figures give the points of the final ζ-plane corresponding to angles of the whole figure.

(α) *Rhombus.*



The equation is

$$z : z-1 : 1 :: \zeta(\zeta-a) : (\zeta-1)^2 : C;$$

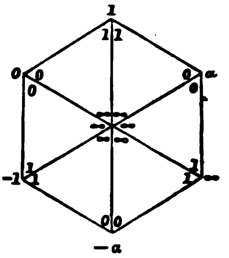
therefore
$$a = 2, \quad C = -1,$$

and the required equation [combining with (ii.)] is

$$4\zeta(2-\zeta) - 2 = \wp' w.$$

(β) *Regular hexagon.*—Here considerations of symmetry will shorten the determination of the constants.

The figure may clearly be taken thus :—



and the equation will be

$$z : z-1 : 1 :: \zeta^2(\zeta^2-a^2)^2 : C(\zeta^2-1)^3 : (\zeta^2+\gamma^2)^3 ;$$

therefore $$a^2 = 10+3\sqrt{5}, \qquad \delta^2 = 3+2\sqrt{5},$$

$$C = -(29+12\sqrt{5}) ;$$

and these, with $$2(2z-1) = \wp' w,$$

complete the representation. In this case it is clear that the form of the resulting equation would be simpler were the hexagon represented on a circle instead of a half-plane. There is little difficulty in showing that the required equation then is

$$\zeta^3 + \frac{1}{\zeta^3} = \wp' w.^*$$

(γ) *A two-sheeted equilateral triangle with the winding point at its centre.*—This figure may be formed by the juxtaposition of six triangles $\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3}\right)$, the obtuse angles all coinciding and continuity being assumed between the first side of the first triangle and the last side of the sixth. If, in this case, the figure in the ζ-plane be taken a unit circle, centre the origin, instead of a half-plane, the necessary equation may, from considerations of symmetry, at once be written in the form

$$z : z-1 : 1 :: (\zeta^3+1)^2 : 4\zeta^3 : (\zeta^3-1)^2,$$

which, with $$z + \frac{1}{z} + 2 = \wp'^2 w \quad \text{[from (ii.)]},$$

completes the representation.

* This case is given by Schartz (*Gesammelte Abhandlungen*, II., p. 252); he obtains it by the transformation of the corresponding integral.

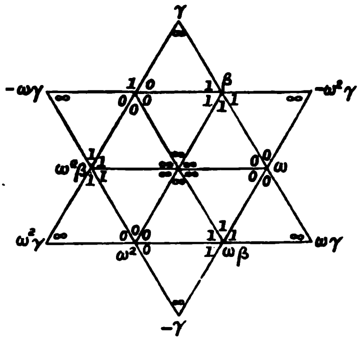
As a verification, these equations give

$$4^{\frac{1}{3}}.3\,dw = \frac{dz}{z^{\frac{5}{6}}(z-1)^{\frac{1}{3}}}, \quad dz = -12\,\frac{\zeta^3+1}{(\zeta^3-1)^3}\,\zeta^2 d\zeta;$$

therefore
$$-4^{-\frac{1}{3}}dw = \frac{\zeta\, d\zeta}{(\zeta^6-1)^{\frac{2}{3}}},$$

so that the figure in the w-plane has six angles, each of $\frac{\pi}{3}$, and a simple winding point within it, as it should have.

(δ) *A six-rayed star.*



$\omega^3 = 1.$

If the figure be represented on a unit circle in the ζ-plane, the symmetry round the centre shows that the correspondence of angles and points indicated in the figure is a possible one. The equation will be

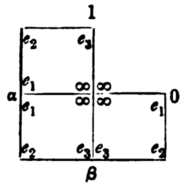
$$z : z-1 : 1 :: (\zeta^3-1)^4 : (\zeta^3-\beta^3)^4 : C\zeta^3(\zeta^6-\gamma^6);$$

therefore
$$\beta^3 = -1, \quad \gamma^6 = -1, \quad C = -8,$$

and
$$\frac{(\zeta^3-1)^4}{2\zeta^3(\zeta^6+1)} + 2 + \wp' w = 0$$

gives the representation.

(ϵ) The simplest case arising from a rectangle, other than those leading to the formulæ for the transformation of elliptic functions, is that indicated in the following figure:—



The corresponding equation must be of the form

$$z-e_1 : z-e_3 : e_3-e_1 :: \zeta(\zeta-a)^2 : (\zeta-1)(\zeta-\beta)^2 : C,$$

$z = e_2$ being no branch-point of ζ.

The constants are at once found to be

$$a = \tfrac{3}{4}, \quad \beta = \tfrac{1}{4}, \quad C = \tfrac{1}{16},$$

and hence the equation effecting the representation is

$$\frac{\wp w - e_1}{e_3 - e_1} = \zeta(4\zeta-3)^2.$$

(ζ) As a last example it may be shown, in a manner similar to that used in the third, that a two-sheeted rectangle with a branch-point at its centre is represented conformally on the surface of a unit circle by the equation

$$\wp w - e_1 : \wp w - e_3 : e_3 - e_1 :: (\zeta^4+1)^2 : (\zeta^4-1)^2 : 4\zeta^4.$$

5. *Conformal Representation of Multiply-connected Areas.*

If the area formed from triangles or rectangles be not simply-connected, it is impossible to represent it conformally on a half-plane, but still a process closely parallel to that which has been explained will give the representation on a standard area of the same multiplicity of connection as the given area. Suppose, for instance, that the area formed from N constituent triangles is doubly-connected. The N half z-planes will then form one (symmetrical) half of a doubly-connected Riemann's surface with definite branch-points at 0, 1, ∞, and at these points only. On such a surface there is no function of position which takes every value once only, but there are necessarily functions taking every value only twice.

If ζ is such a function it must be connected with z by an equation of the form

$$z^2 f_3(\zeta) + z f_2(\zeta) + f_1(\zeta_1) = 0 \quad \text{.....................(i.)},$$

where one at least of the functions f_1, f_2, f_3 must be of degree N. An inspection of the arrangement of the N triangles in the given figure *will determine*, as in the case of a simply-connected area, the nature

of the three branch-points, so that the forms of f_1, f_2, f_3 will be given by

$$f_1(\zeta) = A\prod_1^r(\zeta-\alpha_1)^{m_1},$$

$$f_3(\zeta) = C\prod_1^t(\zeta-\gamma_1)^{p_1},$$

$$f_1(\zeta)+f_2(\zeta)+f_3(\zeta) = B\prod_1^s(\zeta-\beta_1)^{n_1},$$

where the indices m, n, p are determined by inspection, and where, of the constants α, β, γ, two, but not more, may be infinity, in which case the corresponding factors are replaced by unity.

Assuming for simplicity that ζ does not become infinite at any of the branch-points, it follows that

$$\sum_1^r m_1 = \sum_1^s n_1 = \sum_1^t p_1 = N,$$

and $$\sum_1^r(m_1-1)+\sum_1^s(n_1-1)+\sum_1^t(p_1-1) = 2N,$$

so that $$r+s+t = N.$$

Lastly, since the only irrationality that z contains is the square root of a quartic function, the expression

$$\left[f_2(\zeta)\right]^2-4f_1(\zeta)f_3(\zeta),$$

must have a square factor of degree $N-2$.

This is equivalent to $N-1$ equations of condition. Now, since the ratios only of A, B, C occur, the functions f_1, f_2, f_3 contain $N+2$ arbitrary constants, and hence, when three of these (say the two points where ζ becomes infinite and one of the points where it becomes zero) are assigned, the determination of the constants becomes definite. When the area to be represented is given, equation (i.) can therefore be obtained by algebraical processes, and, in the symmetrical half of the corresponding surface on which the area was originally represented, ζ will take every value once, and once only. Hence, when (i.) is combined with the equation between z and $\wp w$, the resulting equation between ζ and $\wp w$ represents the area conformally on an infinite plane, rendered doubly-connected by two slits regarded as boundaries.

When the area has a line of symmetry, the process referred to at the end of § 3 may be taken advantage of to simplify its representation on a slit infinite plane. Thus, if the figure there given is represented on a half-plane by an equation

$$f\left[\zeta, \wp(w)\right] = 0,$$

so that the points M, N, D, E of the boundary correspond to points m, n, d, e of the real axis in the ζ-plane, the same equation will represent the doubly-connected figure formed of twelve rectangles, obtained by reflecting the given figure in the line of the sides MN and DE, on an infinite plane the positive and negative halves of which are continuous only along the segments mn and de of the real axis. When this method can be applied, the previous general one clearly cannot, since z and ζ are not independent functions of position on the corresponding Riemann's surface.

As an illustration, the actual equations may be given for the representation of the area between two similar, similarly situated, and concentric squares, the larger of which is four times the smaller on a slit plane.

Using the figure at the end of § 3 in conjunction with example (ϵ), e_1, e_2, e_3 are -1, 0, 1 when the rectangles are squares, and the equation connected with the figure $ABIJKLMN$ is

$$z+1 : z-1 : 2 :: \zeta(4\zeta-3)^2 : (\zeta-1)(4\zeta-1)^2 : 1.$$

The values of ζ at B, N, A are $0, 1, \infty$, and its values at I, K, M must be in ascending order of magnitude, while they correspond to $z=0$, or to

$$1 = 2\zeta(4\zeta-3)^2.$$

They are therefore

$$\frac{2-\sqrt{3}}{4}, \quad \frac{1}{2}, \quad \frac{2+\sqrt{3}}{4}.$$

The equation connecting this figure and the complete figure of § 3 is therefore

$$\zeta'^2 = C\,\frac{\zeta}{\zeta - \frac{2-\sqrt{3}}{4}},$$

where C may be chosen arbitrarily.

If C be taken unity the values of ζ' are N, D, and at M, E, where ζ is 1 and $\frac{2+\sqrt{3}}{4}$ respectively, are $\pm\frac{2\sqrt{2}}{\sqrt{3}+1}$ and $\pm\frac{\sqrt{3}+1}{2 \cdot 3^{\frac{1}{4}}}$.

Hence, finally,

$$\wp w + 1 = \frac{\left[(\sqrt{3}-1)\zeta'\{(\sqrt{3}+1)\zeta'^2-3\}\right]^2}{(\zeta'^2-1)^3}$$

represents the area on an infinite plane bounded by straight slits

from $\dfrac{-2\sqrt{2}}{\sqrt{3}+1}$ to $\dfrac{2\sqrt{2}}{\sqrt{3}+1}$, and from $-\dfrac{\sqrt{3}+1}{2.3^{\frac{1}{4}}}$ through infinity to $\dfrac{\sqrt{3}+1}{2.3^{\frac{1}{4}}}$.

6. *Conclusion.*

Returning to the case in which the given area is simply-connected, it is interesting to consider the equation

$$f(z, \wp w) = 0,$$

which effects the representation from the point of view of Riemann's theory. The equation is of finite degree N in z, and transcendental in w. If, then, the surface be regarded as extended over the w-plane, it will be N-sheeted, while z is a function which takes every value an infinite number of times on the surface. If N is greater than unity, the surface is of an infinitely high degree of connectivity, but the branch-points and the distribution of the sheets in connection with each will be perfectly definite. To determine the nature of each branch-point and the possible arrangements of branch-lines, it is sufficient to deal with the value of z in the neighbourhood of each angular point of the original area. Thus, in the equation for the representation of a regular hexagon,

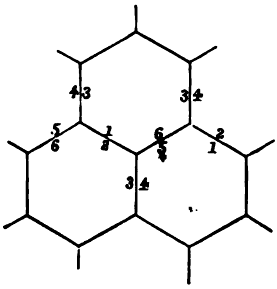
$$z^3 + \frac{1}{z^3} = \wp' w \qquad (g_2 = 0,\ g_3 = -4),$$

the values of $\wp' w$ at three angular points of the hexagon are 2, and at the other three -2. Near one of the first three

$$(z^3 - 1)^2 \propto w - w_0,$$

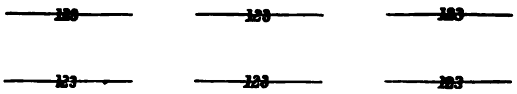
so that the six sheets are connected in pairs, and the same clearly holds for the other three angles.

From each branch-point three simple branch-lines must diverge, each connecting a different pair of sheets. A complete system of branch-lines, ensuring the necessary connection between the sheets, may therefore be represented by the figure



where the numbers indicate the two sheets that are connected by each branch-line.

In the case of figure (ϵ) it is clear that the only branch-points are at the re-entrant angles of the area, and that each of these is a double branch-point connecting three sheets cyclically. A complete system of branch lines will be given by the figure



continued, of course, as in the former case, indefinitely in both directions.

A consideration of the nature of the surface corresponding to the equation

$$f(z, \wp w) = 0$$

becomes of importance, from quite another point of view, when the z is regarded as a particular case of an automorphic function.

The double area obtained by taking together a given area and its reflection in any one of its sides, is a fundamental region for the group of substitutions arising from pairs of reflections in the sides of the given area. But this group is, in general, identical with that arising from pairs of reflections in the side of the constituent triangle or rectangle from which the area has been built up; and the simplest fundamental region is then formed by one pair of the triangles or rectangles. This fundamental region can be within certain limits altered in form (for the simple class of groups here under consideration it cannot be altered in area), but what is here made clear, by a simple instance, is that it depends entirely on the nature of the continuum in which it is to be taken, and that generally when a region is given whose sides are connected in pairs by substitutions belonging to the group (from which the complete group can be constructed) a continuum can be found on which it is a fundamental region.

Thursday, May 11th, 1893.

A. B. KEMPE, Esq., F.R.S., President, in the Chair.

Señor Tamborrel, of Mexico, was elected a member, and Mr. A. L. Dixon was admitted into the Society.

The following communications were made:—

On the Collapse of Boiler Flues: Mr. A. E. H. Love.

On some Formulæ of Codazzi and Weingarten in relation to the application of Surfaces to each other: Prof. Cayley.

On the Expansion of certain Infinite Products: Prof. L. J. Rogers.

A Theorem for Bicircular Quartic Curves and for Cyclides, analogous to Ivory's Theorem for Conics and Conicoids: Mr. A. L. Dixon.

On the Linear Transformations between two Quadrics: Mr. H. Taber.

On Maps and the Problem of the Four Colours: Prince C. de Polignac.

On Fermat's Proof that Primes of the Form $4n+1$ can be broken up into the Sum of Two Squares: Mr. S. Roberts.

Supplementary Note on Complex Primes formed with the Fifth Roots of Unity: Prof. Lloyd Tanner.

The following presents were received:—

A Cabinet Likeness of Mr. Love, presented by Mr. Love.

"Voordrachten over den Grondslag van een Bibliographisch Repertorium der Wiskundige Wetenschappen," E and M² 3.

"Beiblätter zu den Annalen der Physik und Chemie," Band XVII., Stück 4.

"Proceedings of the Royal Society," Vol. LII., No. 320.

Lemoine, M. E.—"Application d'une Méthode d'Evaluation de la Simplicité des Constructions à la Comparaison de quelques Solutions du Problème d'Apollonius." (Extrait des *Nouvelles Annales de Mathématiques.*)

Lemoine, M. E.—"Application de la Géométrographie à l'examen de diverses Solutions d'un même Problème." (Extrait du *Bulletin de la Société Mathématique de France.*)

Lemoine, M. E.—"Résultats et Théorèmes divers concernant la Géométrie du Triangle, &c." (Congrès de Pau, 1892.)

"Nyt Tidsskrift for Mathematik," A. Fjerde Aargang, Nos. 1, 2; Copenhagen.

"Nyt Tidsskrift for Mathematik," B. Fjerde Aargang, No. 1; Copenhagen.

"Transactions of the Connecticut Academy of Arts and Sciences," Vol. VIII., Part 2, 1893, and Vol. IX., Part 1; Newhaven, 1892.

"Bulletin of the New York Mathematical Society," Vol. ii., No. 7; April, 1893.

"Index du Répertoire Bibliographique des Sciences Mathématiques," Paris, 1893.

"Bulletin des Sciences Mathématiques," 2ème Serie, Tome xvii.; Mars, 1893.

"Bulletin de la Société Mathématique de France," Tome xxi., No. 3; Paris.

"Rendiconti del Circolo Matematico di Palermo," Fasc. 1 and 2, Tomo vii.; 1893.

"Annales de la Faculté des Sciences de Toulouse," Fasc. 1, Tome vii., 1893; Paris.

"Atti della Reale Accademia dei Lincei, 5° Serie — Rendiconti," Vol. ii., Fasc. 6, Sem. 1; Roma, 1893.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," xli.–lv., Title page, Contents, and Index, for 1892.

"Annali di Matematica," Serie ii., Tomo xxi., Fasc. 1, 1893; Milano.

"Journal für die reine und angewandte Mathematik," Band cxi., Heft 3, 1893; Berlin.

"Educational Times," May, 1893.

"Indian Engineering," Vol. xiii., Nos. 12–15.

On the Collapse of Boiler Flues. *By* A. E. H. Love.

Read May 11th, 1893.

Abstract.

The problem consists in discovering the conditions of a collapse of a thin cylindrical shell under external pressure, when the ends are constrained to occupy fixed positions. Since all problems of collapse depend on the geometrical possibility of finite displacements being accompanied by only infinitesimal strains, it appears at the outset that, unless the shell can receive a displacement of pure bending without stretching of the middle surface, collapse is impossible. The assumed condition of no terminal displacement is equivalent to closing the ends of the shell, and, since a closed surface cannot be bent without stretching, this condition apparently precludes the possibility of collapse. On the other hand, it is well known that, if the external pressure exceed a certain value, an infinitely long cylindrical shell of given small thickness and given diameter will collapse under the pressure. The critical pressure has been determined by Bryan, and Basset, who find the same result. It is therefore to be expected that, if the cylinder is of sufficient length, the extensional displace-

ment which must be superposed upon the displacement of pure bending in order to satisfy the end conditions will be practically unimportant except in the neighbourhood of the ends. The problem is thus reduced to discovering the order of magnitude of the length of the shell, in order that it may be treated as infinite when the thickness is small. For this purpose, consider the case where the pressure is just equal to the critical pressure, and the displacement of pure bending in the infinite cylinder is consequently of the form

$$u = 0, \quad v = \tfrac{1}{2}A\cos 2\phi, \quad w = A\sin 2\phi,$$

where A is a small arbitrary constant. (The displacement u is parallel to the generator, v is along the circular section, and w along the radius outwards.) By means of displacements of this form the equations of equilibrium can be satisfied, but the boundary conditions at the ends cannot. Now take the case of an infinite cylinder with an end $x = 0$, at which v and w must vanish, and seek a displacement involving both flexure and extension of the middle surface to be superposed on the displacement given by the above forms, such displacement to satisfy the equations of equilibrium, and the boundary conditions:—(1) that the new v and w are equal and opposite to those above given at $x = 0$, (2) that the new u, v, w vanish at $x = \infty$. The required solution can be determined, and is of the form

$$u = e^{-mx}(A_1\cos mx + B_1\sin mx)\sin 2\phi,$$

$$v = e^{-mx}(A_2\cos mx + B_2\sin mx)\cos 2\phi,$$

$$w = e^{-mx}\frac{m^2d^2}{4(2+\sigma)}(B_2\cos mx - A_2\sin mx)\sin 2\phi,$$

in which B_2 and A_2 can be determined so as to satisfy the conditions at $x = 0$, σ is the Poisson's ratio of the material of the shell, and

$$m = \left[12(1-\sigma^2)\right]^{\frac{1}{4}}/\sqrt{(dt)},$$

t is the thickness, and d the diameter of the shell. If σ be taken equal to $\frac{1}{4}$, the reciprocal of m is about ·546 of the mean proportional between the thickness and the diameter, and it follows that whenever x is great compared with this quantity the influence of the end is unimportant, and the displacement approximates to one of pure bending. To make the tendency to collapse occur in practice, it would be necessary that the half-length of the flue should be great compared

with m^{-1}, and the practical conclusion would be that for a flue of length l stability would be secured if

$$\tfrac{1}{2}l < n/m, \quad \text{or} \quad l < N\sqrt{(dt)},$$

where N is a considerable number. It is customary in stationary boilers to make the flues in detached pieces connected by massive flanged joints, so that the effective length of the flue is the distance between consecutive joints. If the number N be taken equal to 12, we have the rule that the distance between the joints must be not greater than twelve times the mean proportional between the thickness and the diameter. The value $N = 12$ accords well with what has been found safe in practice, but the rule as to spacing the joints is new.

On some Formulæ of Codazzi and Weingarten in relation to the application of Surfaces to each other. By Prof. Cayley.

Received May 1st, 1893. Read May 11th, 1893.

An extremely elegant theory of the application of surfaces one upon another is developed in the memoir, Codazzi, "Mémoire relatif à l'application des surfaces les unes sur les autres," *Mem. Pres. de l'Institut*, t. XXVII. (1883), No. 6, pp. 1–47; but the notation is not presented in a form which is easily comparable with that of the Gaussian notation in the theory of surfaces. I propose to reproduce the theory in the Gaussian notation.

Codazzi considers on a given surface two systems of curves depending on the parameters t, T respectively; the curves are in the memoir taken to be orthogonal to each other, but this restriction is removed in the general formula given p. 44, *Addition au Chapitre premier*. For a curve of either system, he considers the tangent, the principal normal, or normal in the osculating plane, and the binormal, or line at right angles to the osculating plane (say these are tan, prn and bin). For a curve of the one system, that in which t is variable

or say a t-curve, he denotes the cosine-inclinations of these lines to the axes by the letters a, b, c, thus :

	x	y	z
tan	a_x	a_y	a_z
prn	b_x	b_y	b_z
bin	c_x	c_y	c_z

and he writes also l for the inclination of the principal normal to the normal of the surface, $\dfrac{dm}{dt}\,dt$ for the angle of contingence, or inclination of the tangent at the point (t, T) to the tangent at the point $(t+dt, T)$, and $\dfrac{dn}{dt}\,dt$ for the angle of torsion, or inclination of the osculating plane at the point (t, T) to that at the point $(t+dt, T)$, or, what is the same thing, the inclination of the binormals at these points respectively. And he gives as known formulæ

$$\frac{da}{dt}=b\,\frac{dm}{dt},\quad \frac{db}{dt}=-a\,\frac{dm}{dt}-c\,\frac{dn}{dt},\quad \frac{dc}{dt}=b\,\frac{dn}{dt},$$

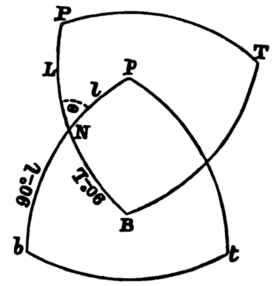
where a, b, c denote

$$(a_x,\, b_x,\, c_x),\quad (a_y,\, b_y,\, c_y),\quad \text{or}\quad (a_z,\, b_z,\, c_z).$$

He uses the capital letters

$$A_x,\, A_y,\, A_z,\, B_x,\, B_y,\, B_z,\, C_x,\, C_y,\, C_z,\, L,\, \frac{dM}{dT}\,dT,\, \frac{dN}{dT}\,dT$$

with the like significations in regard to the curve for which T is variable, or say the T-curve; for greater clearness I give the accompanying figure.



Codazzi writes further

$$\frac{dm}{dt}\cos l = u, \quad \frac{dm}{dt}\sin l = v, \quad \frac{dn}{dt}-\frac{dl}{dt} = w,$$

$$\frac{dM}{dT}\cos L = U, \quad \frac{dM}{dT}\sin L = V, \quad \frac{dN}{dT}-\frac{dL}{dT} = W;$$

also, if s, S are the arcs of the two curves respectively,

$$\frac{ds}{dt} = r, \quad \frac{dS}{dT} = R;$$

and he obtains a system of six formulæ, which in the *Addition*, p. 44, are presented in the following form—only I use therein θ, instead of his b, to denote the inclination of the two curves to each other:

$$\frac{du}{dT} = \frac{dU}{dt}\cos\theta + \frac{dW}{dt}\sin\theta + w\left(V-\frac{d\theta}{dT}\right) + (U\sin\theta - W\cos\theta)\left(v-\frac{d\theta}{dt}\right),$$

$$\frac{dU}{dt} = \frac{du}{dT}\cos\theta + \frac{dw}{dT}\sin\theta + W\left(v-\frac{d\theta}{dt}\right) + (u\sin\theta - w\cos\theta)\left(V-\frac{d\theta}{dT}\right),$$

$$\left(\frac{dv}{dT}+\frac{dV}{dt}\right)\frac{d^2\theta}{dt\,dT} + \sin\theta\,(uU-wW) - \cos\theta\,(uW+wU) = 0,$$

$$R\,(u\cos\theta + w\sin\theta) = r\,(U\cos\theta + W\sin\theta),$$

$$R\sin\theta\left(v-\frac{d\theta}{dt}\right) + \frac{dR}{dt}\cos\theta = \frac{dr}{dT},$$

$$r\sin\theta\left(V-\frac{d\theta}{dT}\right) + \frac{dr}{dT}\cos\theta = \frac{dR}{dt}.$$

In the Gaussian notation, taking p, q for the parameters, we have, with the slight variations presently referred to,

$$dx = a\,dp + a'dq + \tfrac{1}{2}\alpha\,dp^2 + \alpha' dp\,dq + \alpha'' dq^2,$$

$$dy = b\,dp + b'dq + \tfrac{1}{2}\beta\,dp^2 + \beta' dp\,dq + \beta'' dq^2,$$

$$dz = c\,dp + c'dq + \tfrac{1}{2}\gamma\,dp^2 + \gamma' dp\,dq + \gamma'' dq^2,$$

$$A,\ B,\ C = bc'-b'c,\ ca'-c'a,\ ab'-a'b,$$

$$E,\ F,\ G = a^2+b^2+c^2,\ aa'+bb'+cc',\ a'^2+b'^2+c'^2;$$

and therefore $dx^2+dy^2+dz^2 = Edp^2+2Fdpdq+Gdq^2$;

$$\sqrt{EG-F^2} = V;$$

$$E',\ F',\ G' = A\alpha+B\beta+C\gamma,\quad A\alpha'+B\beta'+C\gamma',\quad A\alpha''+B\beta''+C\gamma'';$$

and I take further

$$\begin{array}{llll}
\omega,\ \omega',\ \omega'' = & a\alpha+b\beta+c\gamma, & a\alpha'+b\beta'+c\gamma', & a\alpha''+b\beta''+c\gamma'', \\
\varpi,\ \varpi',\ \varpi'' = & a'\alpha+b'\beta+c'\gamma, & a'\alpha'+b'\beta'+c'\gamma', & a'\alpha''+b'\beta''+c'\gamma'', \\
\lambda,\ \lambda',\ \lambda'' = & \alpha^2+\beta^2+\gamma^2, & \alpha'^2+\beta'^2+\gamma'^2, & \alpha''^2+\beta''^2+\gamma''^2, \\
\mu,\ \mu',\ \mu'' = & \alpha'\alpha''+\beta'\beta''+\gamma'\gamma'', & \alpha''\alpha+\beta''\beta+\gamma''\gamma, & \alpha\alpha'+\beta\beta'+\gamma\gamma',
\end{array}$$

$$E\lambda-\omega^2 = \Delta,\quad G\lambda''-\varpi''^2 = \Delta'',$$

where it is to be noticed that V^2, E', F', G', are written instead of Gauss's Δ, D, D', D'', and ω, ω', ω'', ϖ, ϖ', ϖ'' instead of his m, m', m'', n, n', n''; and that he gives for the last mentioned quantities the values

$$m = \tfrac{1}{2}\frac{dE}{dp},\quad m' = \tfrac{1}{2}\frac{dE}{dq},\quad m'' = \frac{dF}{dq}-\tfrac{1}{2}\frac{dG}{dp},$$

$$n = \frac{dF}{dp}-\tfrac{1}{2}\frac{dE}{dq},\quad n' = \tfrac{1}{2}\frac{dG}{dp},\quad n'' = \tfrac{1}{2}\frac{dG}{dq},$$

or, say
$$\omega = \tfrac{1}{2}E_1,\quad \omega' = \tfrac{1}{2}E_2,\quad \omega'' = F_2-\tfrac{1}{2}G_1,$$

$$\varpi = F_1-\tfrac{1}{2}E_2,\quad \varpi' = \tfrac{1}{2}G_1,\quad \varpi'' = \tfrac{1}{2}G_2,$$

where the subscripts (1) and (2) denote differentiation in regard to p and q respectively.

Observing that the cosine-inclinations of the tangent to the p-curve are as a, b, c, and those of the binormal or perpendicular to the osculating plane are as $b\gamma-c\beta$, $c\alpha-a\gamma$, $a\beta-b\alpha$, we easily find

$$a_x,\ a_y,\ a_z = \frac{a}{\sqrt{E}},\quad \frac{b}{\sqrt{E}},\quad \frac{c}{\sqrt{E}},$$

$$c_x,\ c_y,\ c_z = \frac{b\gamma-c\beta}{\sqrt{\Delta}},\quad \frac{c\alpha-a\gamma}{\sqrt{\Delta}},\quad \frac{a\beta-b\alpha}{\sqrt{\Delta}},$$

$$b_x,\ b_y,\ b_z = \frac{E\alpha-a\omega}{\sqrt{E\Delta}},\quad \frac{E\beta-b\omega}{\sqrt{E\Delta}},\quad \frac{E\gamma-c\omega}{\sqrt{E\Delta}},$$

and, for the cosine-inclinations of the normal of the surface

$$\Delta_x,\ \Delta_y,\ \Delta_z = \frac{A}{V},\ \frac{B}{V},\ \frac{C}{V},$$

we find

$$\cos l = \frac{A\,(b\gamma - c\beta) + B\,(c\alpha - a\gamma) + C\,(a\beta - b\alpha)}{V\sqrt{\Delta}}$$

$$= \frac{(a^2+b^2+c^2)(a'\alpha+b'\beta+c'\gamma) - (aa'+bb'+cc')(a\alpha+b\beta+c\gamma)}{V\sqrt{\Delta}}$$

$$= \frac{E\varpi - F\omega}{V\sqrt{\Delta}},$$

$$\sin l = \frac{A\,(E\alpha - a\omega) + B\,(E\beta - b\omega) + C\,(E\gamma - c\omega)}{V\sqrt{E\Delta}};$$

or, since $Aa + Bb + Cc = 0$, this is

$$\sin l = \frac{E\,(A\alpha + B\beta + C\gamma)}{V\sqrt{E\Delta}} = \frac{E'\sqrt{E}}{V\sqrt{\Delta}}.$$

We ought therefore to have

$$E'^2E + (E\varpi - F\omega)^2 = V^2\Delta, = (EG - F^2)(E\lambda - \omega^2);$$

or, omitting the terms $F^2\omega^2$, which destroy each other, and throwing out a factor E, this is

$$E\varpi^2 - 2F\varpi\omega + G\omega^2 - \lambda\,(EG - F^2) = \lambda\,(EG - F^2) - E'^2;$$

viz., putting for $EG - F^2$ its value, $= A^2 + B^2 + C^2$, and for the other terms their values, this is

$$(a^2+b^2+c^2)(a'\alpha+b'\beta+c'\gamma)^2$$
$$-2\,(aa'+bb'+cc')(a'\alpha+b'\beta+c'\gamma)(a\alpha+b\beta+c\gamma)$$
$$+(a'^2+b'^2+c'^2)(\alpha^2+\beta^2+\gamma^2)$$
$$= (A^2+B^2+C^2)(\alpha^2+\beta^2+\gamma^2) - (A\alpha+B\beta+C\gamma)^2.$$

The right-hand side is here $= (B\gamma - C\beta)^2 + (C\alpha - A\gamma)^2 + (A\beta - B\alpha)^2$; the left-hand consists of three parts, the first whereof is

$$\{a\,(a'\alpha+b'\beta+c'\gamma) - a'\,(a\alpha+b\beta+c\gamma)\}^2 = (-B\gamma + C\beta)^2,$$

and similarly the other two parts are

$$(-C\alpha + A\gamma)^2 \quad\text{and}\quad (-A\beta + B\alpha)^2;$$

the equation is thus verified.

We require Codazzi's $\frac{dm}{dt}$ and $\frac{dn}{dt}$, or say $\frac{dm}{dp}$ and $\frac{dn}{dp}$; these are to be obtained from the equations

$$\frac{d}{dp}\frac{\alpha}{\sqrt{E}} = \frac{E\alpha - a\omega}{\sqrt{E\Delta}}\frac{dm}{dp},\quad \frac{d}{dp}\frac{b\gamma - c\beta}{\sqrt{\Delta}} = \frac{E\alpha - a\omega}{\sqrt{E\Delta}}\frac{dn}{dp},$$

and the values obtained should satisfy

$$\frac{d}{dp}\,\frac{Ea-a\omega}{\sqrt{E\Delta}}=-\frac{a}{\sqrt{E}}\,\frac{dm}{dp}-\frac{b\gamma-c\beta}{\sqrt{\Delta}}\,\frac{dn}{dp}.$$

I find
$$\frac{dm}{dp}=\frac{\sqrt{\Delta}}{E},\quad \frac{dn}{dp}=-\frac{\sqrt{E}}{\Delta}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix},$$

where $\alpha_1, \beta_1, \gamma_1$ are the derivatives of α, β, γ in respect to p; and the equation to be verified thus is

$$\frac{d}{dp}\,\frac{Ea-a\omega}{\sqrt{E\Delta}}=-\frac{a\sqrt{\Delta}}{E\sqrt{E}}+\frac{(b\gamma-c\beta)\sqrt{E}}{\Delta\sqrt{\Delta}}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

First, for $\dfrac{dm}{dp}$: the derivatives of a, E are α and $2\,(a\alpha+b\beta+c\gamma)$, $=2\omega$; we thus have

$$\frac{d}{dp}\,\frac{a}{\sqrt{E}},=\frac{\alpha}{\sqrt{E}}-\frac{a\omega}{E\sqrt{E}}=\frac{E\alpha-a\omega}{E\sqrt{E}},$$

which is
$$=\frac{E\alpha-a\omega}{\sqrt{E\Delta}}\,\frac{dm}{dp};$$

viz., we have $\dfrac{dm}{dp}=\dfrac{\sqrt{\Delta}}{E}$.

Next, for $\dfrac{dn}{dp}$: using a subscript ($_1$) to denote derivation in regard to p, we have $\Delta=E\lambda-\omega^2$, and thence

$$\begin{aligned}\Delta_1&=E_1\lambda+E\lambda_1-2\omega\omega_1\\ &=2\omega\lambda+E.2\,(\alpha\alpha_1+\beta\beta_1+\gamma\gamma_1)-2\omega(\lambda+a\alpha_1+b\beta_1+c\gamma_1),\\ &=2\left\{E\,(\alpha\alpha_1+\beta\beta_1+\gamma\gamma_1)-\omega\,(a\alpha_1+b\beta_1+c\gamma_1)\right\},\\ &=2\,(\alpha_1X+\beta_1Y+\gamma_1Z),\end{aligned}$$

if, for a moment,

$$X,\ Y,\ Z=E\alpha-a\omega,\ E\beta-b\omega,\ E\gamma-c\omega;$$

these values give identically

$$\alpha X+\beta Y+\gamma Z=\Delta,\quad\text{and}\quad aX+bY+cZ=0.$$

Hence we have

$$\frac{d}{dp}\,\frac{b\gamma-c\beta}{\sqrt{\Delta}}=\frac{b\gamma_1-c\beta_1}{\sqrt{\Delta}}-\frac{(b\gamma-c\beta)(\alpha_1X+\beta_1Y+\gamma_1Z)}{\Delta\sqrt{\Delta}},$$

which must be $\quad =\dfrac{X}{\sqrt{E\Delta}}\,\dfrac{dn}{dp}$;

we thus have

$$(b\gamma_1-c\beta_1)\,\Delta-(b\gamma-c\beta)(\alpha_1X+\beta_1Y+\gamma_1Z)=\frac{\Delta X}{\sqrt{E}}\,\frac{dn}{dp};$$

or, putting the left-hand side in the form

$$\begin{aligned}&(b\gamma_1-c\beta_1)(\alpha X+\beta Y+\gamma Z)\\-&(b\gamma-c\beta)(\alpha_1X+\beta_1Y+\gamma_1Z)\\-&(\beta\gamma_1-\beta_1\gamma)(aX+bY+cZ),\end{aligned}$$

this is $\quad =-X\begin{vmatrix}a, & b, & c\\ \alpha, & \beta, & \gamma\\ \alpha_1, & \beta_1, & \gamma_1\end{vmatrix},$

and we thus find $\quad \dfrac{dn}{dp}=-\dfrac{\sqrt{E}}{\Delta}\begin{vmatrix}a, & b, & c\\ \alpha, & \beta, & \gamma\\ \alpha_1, & \beta_1, & \gamma_1\end{vmatrix}.$

For the verification of the equation

$$\frac{d}{dp}\,\frac{E\alpha-a\omega}{\sqrt{E\Delta}}=-\frac{\alpha\sqrt{\Delta}}{E\sqrt{E}}+\frac{(b\gamma-c\beta)\sqrt{E}}{\Delta\sqrt{\Delta}}\begin{vmatrix}a, & b, & c\\ \alpha, & \beta, & \gamma\\ \alpha_1, & \beta_1, & \gamma_1\end{vmatrix},$$

we have $\quad (E\alpha-a\omega)_1=E\alpha_1-a\,(a\alpha_1+b\beta_1+c\gamma_1)+\alpha\omega-a\lambda,$

$$(E\Delta)_1=2E^2\,(\alpha\alpha_1+\beta\beta_1+\gamma\gamma_1)-2E\omega\,(a\alpha_1+b\beta_1+c\gamma_1)+2\Delta\omega,$$

and hence the equation is

$$-\frac{\{E^2\,(\alpha\alpha_1+\beta\beta_1+\gamma\gamma_1)-E\omega\,(a\alpha_1+b\beta_1+c\gamma_1)+\omega\}\,(E\alpha-a\omega)}{E\Delta\sqrt{E\Delta}}$$

$$+\frac{E\alpha_1-a\,(a\alpha_1+b\beta_1+c\gamma_1)+\alpha\omega-a\lambda}{\sqrt{E\Delta}}$$

$$=-\frac{\alpha\sqrt{\Delta}}{E\sqrt{E}}+\frac{(b\gamma-c\beta)\sqrt{E}}{\Delta\sqrt{\Delta}}\begin{vmatrix}a, & b, & c\\ \alpha, & \beta, & \gamma\\ \alpha_1, & \beta_1, & \gamma_1\end{vmatrix}.$$

Considering first the terms without $\alpha_1, \beta_1, \gamma_1$, these give

$$-\omega\,(E\alpha - a\omega) + E\,(a\omega - \alpha\lambda) = -a\,(E\lambda - \omega^2),$$

which is identically true; and then the remaining terms with $\alpha_1, \beta_1, \gamma_1$ give

$$-\left\{E\,(\alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1) - \omega\,(a\alpha_1 + b\beta_1 + c\gamma_1)\right\}(E\alpha - a\omega)$$
$$+\left\{E\alpha_1 - a\,(a\alpha_1 + b\beta_1 + c\gamma_1)\right\}\,(E\lambda - \omega^2)$$
$$= (b\gamma - c\beta)\,E \begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

On the left-hand side the whole coefficient of α_1 is

$$= -(E\alpha - a\omega)^2 + (b^2 + c^2)(E\lambda - \omega^2),$$

which is $= -E^2\alpha^2 + 2a\alpha E\omega - a^2\omega^2 + (b^2 + c^2)\,E\lambda - (b^2 + c^2)\,\omega^2,$

$$= E\left[-E\alpha^2 + 2a\alpha\omega + (b^2 + c^2)\,\lambda - \omega^2\right];$$

and, substituting for E, ω, and λ their values, this is found to be

$$= E\left\{(b^2 + c^2)(\beta^2 + \gamma^2) - (b\beta + c\gamma)^2\right\}, = E\,(b\gamma - c\beta)^2.$$

Similarly the whole coefficients of β_1 and γ_1 are found to be

$$= E\,(b\gamma - c\beta)(c\alpha - a\gamma), \text{ and } E\,(b\gamma - c\beta)(a\beta - b\alpha), \text{ respectively,}$$

and thus the left-hand side becomes

$$= (b\gamma - c\beta)\,E \begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix},$$

as it should do.

We have $$\tan l = \frac{P}{Q},$$

where $P = E'\sqrt{E} = \sqrt{E}\,(A\alpha + B\beta + C\gamma), \ = \mathrm{a}\alpha + \mathrm{b}\beta + \mathrm{c}\gamma$, suppose,

$$Q = E\varpi - F\omega = (cB - bC)\,\alpha + (aC - cA)\,\beta + (bA - aB)\,\gamma,$$
$$= \mathrm{a}'\alpha + \mathrm{b}'\beta + \mathrm{c}'\gamma, \text{ suppose,}$$
$$P^2 + Q^2 = V^2\Delta,$$

and we have hence to find $\dfrac{dl}{dp}$. Using, as before, a subscript $(_1)$ to denote derivation in regard to p, we have

$$\frac{dl}{dp} = \frac{QP_1 - PQ_1}{P^2 + Q^2};$$

the numerator is

$$= \quad (\mathrm{a}'\alpha+\mathrm{b}'\beta+\mathrm{c}'\gamma)(\mathrm{a}\alpha_1+\mathrm{b}\beta_1+\mathrm{c}\gamma_1+\mathrm{a}_1\alpha+\mathrm{b}_1\beta+\mathrm{c}_1\gamma)$$
$$-(\mathrm{a}\alpha+\mathrm{b}\beta+\mathrm{c}\gamma)(\mathrm{a}'\alpha_1+\mathrm{b}'\beta_1+\mathrm{c}'\gamma_1+\mathrm{a}_1'\alpha+\mathrm{b}_1'\beta+\mathrm{c}_1'\gamma)$$
$$=-\big[(\mathrm{bc}'-\mathrm{b}'\mathrm{c})(\beta\gamma_1-\beta_1\gamma)+(\mathrm{ca}'-\mathrm{c}'\mathrm{a})(\gamma\alpha_1-\gamma_1\alpha)$$
$$+(\mathrm{ab}'-\mathrm{a}'\mathrm{b})(\alpha\beta_1-\alpha_1\beta)\big]$$
$$+Q\,(\mathrm{a}_1\alpha+\mathrm{b}_1\beta+\mathrm{c}_1\gamma)-P\,(\mathrm{a}_1'\alpha+\mathrm{b}_1'\beta+\mathrm{c}_1'\gamma).$$

For the first part hereof,

$$\mathrm{bc}'-\mathrm{b}'\mathrm{c}=\sqrt{E}\,\{B\,(bA-aB)-C\,(aC-cA)\}$$
$$=\sqrt{E}\,\{A\,(aA+bB+cC)-a\,(A^2+B^2+C^2)\},\ =-aV^2\sqrt{E},$$

since $aA+bB+cC=0$; and similarly

$$\mathrm{ca}'-\mathrm{c}'\mathrm{a},\ \mathrm{ab}'-\mathrm{a}'\mathrm{b}=-\mathrm{b}V^2\sqrt{E},\ \ -\mathrm{c}V^2\sqrt{E},\ \text{respectively};$$

and thus the first part is

$$=V^2\sqrt{E}\,\{a\,(\beta\gamma_1-\beta_1\gamma)+b\,(\gamma\alpha_1-\gamma_1\alpha)+c\,(\alpha\beta_1-\alpha_1\beta)\},$$

$$=V^2\sqrt{E}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$$

Hence, dividing by $\quad P^2+Q^2,\ =V^2\Delta,$

the first part of $\dfrac{dl}{dp}$ is $\quad =\dfrac{\sqrt{E}}{\Delta}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}.$

For the second part of the numerator, we require

$$\alpha\mathrm{a}_1+\beta\mathrm{b}_1+\gamma\mathrm{c}_1 \quad \text{and} \quad \alpha\mathrm{a}_1'+\beta\mathrm{b}_1'+\gamma\mathrm{c}_1';$$

the values of a, b, c are $A\sqrt{E}, B\sqrt{E}, C\sqrt{E}$, where $E_1=2\omega$, and hence

$$\mathrm{a}_1\alpha+\mathrm{b}_1\beta+\mathrm{c}_1\gamma=$$
$$\left(A_1\sqrt{E}+\frac{A\omega}{\sqrt{E}}\right)\alpha+\left(B_1\sqrt{E}+\frac{B\omega}{\sqrt{E}}\right)\beta+\left(C_1\sqrt{E}+\frac{C\omega}{\sqrt{E}}\right)\gamma,$$
$$=\sqrt{E}\,(A_1\alpha+B_1\beta+C_1\gamma)+\frac{\omega}{\sqrt{E}}(A\alpha+B\beta+C\gamma).$$

From the values $A, B, C=bc'-b'c,\ ca'-c'a,\ ab'-a'b,$

we have

$$A_1, B_1, C_1 = \beta c' - \gamma b' + b\gamma' - c\beta',\ \gamma a' - \alpha c' + c\alpha' - a\gamma',\ \alpha b' - \beta a' + a\beta' - b\alpha',$$

and thence

$$A_1\alpha + B_1\beta + C_1\gamma = -\left[a(\beta\gamma' - \beta'\gamma) + b(\gamma\alpha' - \gamma'\alpha) + c(\alpha\beta' - \alpha'\beta)\right],$$

$$= -\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}.$$

Hence $$\mathrm{a}_1\alpha + \mathrm{b}_1\beta + \mathrm{c}_1\gamma = -\sqrt{E}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} + \frac{\omega E'}{\sqrt{E}}.$$

Next, we have $\mathrm{a}', \mathrm{b}', \mathrm{c}' = cB - bC,\ aC - cA,\ bA - aB$,

and thence $\alpha\mathrm{a}_1' + \beta\mathrm{b}_1' + \gamma\mathrm{c}_1'$

$$= \alpha(\gamma B - \beta C + cB_1 - bC_1) + \beta(\alpha C - \gamma A + aC_1 - cA_1)$$
$$+ \gamma(\beta A - \alpha B + bA_1 - aB_1)$$
$$= (b\gamma - c\beta)A_1 + (c\alpha - a\gamma)B_1 + (a\beta - b\alpha)C_1$$
$$= (b\gamma - c\beta)(c'\beta - b'\gamma + b\gamma' - c\beta')$$
$$+ (c\alpha - a\gamma)(a'\gamma - c'\alpha + c\alpha' - a\gamma')$$
$$+ (a\beta - b\alpha)(b'\alpha - a'\beta + a\beta' - b\alpha');$$

the portion hereof which is quadric in α, β, γ is

$$= -(\alpha^2 + \beta^2 + \gamma^2)(aa' + bb' + cc') + (a\alpha + b\beta + c\gamma)(a'\alpha + b'\beta + c'\gamma),$$
$$= -\lambda F + \omega\varpi,$$

and the remaining portion, which is lineo-linear in α, β, γ and α', β', γ', is

$$= (\alpha\alpha' + \beta\beta' + \gamma\gamma')(a^2 + b^2 + c^2) - (a\alpha + b\beta + c\gamma)(a\alpha' + b\beta' + c\gamma'),$$
$$= E\mu'' - \omega\omega';$$

we thus have $\alpha\mathrm{a}_1' + \beta\mathrm{b}_1' + \gamma\mathrm{c}_1' = -\lambda F + E\mu'' + \omega(\varpi - \omega')$.

Hence the second portion of the numerator, or

$$Q(\alpha\mathrm{a}_1 + \beta\mathrm{b}_1 + \gamma\mathrm{c}_1) - P(\alpha\mathrm{a}_1' + \beta\mathrm{b}_1' + \gamma\gamma_1'),$$

is
$$= (E\varpi - F\omega)\left\{-\sqrt{E}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} + \frac{\omega E'}{\sqrt{E}}\right\}$$
$$-E'\sqrt{E}\left\{-\lambda F + \mu'' E + \omega(\varpi - \omega')\right\}.$$

There are two terms, $+E'\sqrt{E}\omega\varpi$ and $-E'\sqrt{E}\omega\varpi$, which destroy each other, and thus the whole second part of the numerator is

$$= -(E\varpi - F\omega)\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} - E'\sqrt{E}(\mu'' E - \lambda F) + \frac{E'\omega}{\sqrt{E}}(\omega' E - \omega F),$$

and, for the corresponding part of $\frac{dl}{dp}$, we must divide this by P^2+Q^2, $= V^2\Delta$.

Hence finally we have

$$\frac{dl}{dp} = \frac{\sqrt{E}}{\Delta}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix} + \frac{1}{V^2\Delta}\left\{-(\varpi E - \omega F)\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix}\right.$$
$$\left. - E'\sqrt{E}(\mu'' E - \lambda F) + \frac{E'\omega}{\sqrt{E}}(\omega' E - \omega F)\right\},$$

where I recall that the values of ω, ϖ, ω', λ, and μ'' are $a\alpha + b\beta + c\gamma$, $a'\alpha + b'\beta + c'\gamma$, $a\alpha' + b\beta' + c\gamma'$, $\alpha^2+\beta^2+\gamma^2$, and $\alpha\alpha' + \beta\beta' + \gamma\gamma'$, respectively. Observe that the first term in this expression for $\frac{dl}{dp}$ is

$$= -\frac{dn}{dp}.$$

We thus have

$$u = \frac{dm}{dp}\cos l = \frac{E\varpi - F\omega}{VE},$$

$$v = \frac{dm}{dp}\sin l = \frac{E'}{V\sqrt{E}},$$

$$w = \frac{dn}{dp} - \frac{dl}{dp} = -\frac{2\sqrt{E}}{\sqrt{\Delta}}\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha_1, & \beta_1, & \gamma_1 \end{vmatrix}$$

$$+\frac{1}{V^2\Delta}\left\{(\varpi E - \omega F)\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} + E'\sqrt{E}(\mu'' E - \lambda F) - \frac{E'\omega}{\sqrt{E}}(\omega' E - \omega F)\right\},$$

and we thence obtain at once the values of U, V, W; viz., these are

$$U = \frac{dM}{dq}\cos L = \frac{G\omega'' - F\varpi''}{VG},$$

$$V = \frac{dM}{dq}\sin L = \frac{G'}{V\sqrt{G}},$$

$$W = \frac{dN}{dq} - \frac{dL}{dq} = -\frac{2\sqrt{G}}{\sqrt{\Delta''}}$$

$$+\frac{1}{V^2\Delta''}\left\{(\omega''G - \varpi''F)\begin{vmatrix} a', & b', & c' \\ \alpha'', & \beta'', & \gamma'' \\ \alpha_2'', & \beta_2'', & \gamma_2'' \end{vmatrix} + G'\sqrt{G}\,(\mu G - \lambda''F) - \frac{G'\varpi''}{\sqrt{G}}(\varpi'G - \varpi''F)\right\},$$

where α_2'', β_2'', γ_2'' denote the derived functions of α'', β'', γ'' in regard to q, viz., these are the third derived functions of x, y, z in regard to q.

We have, moreover,

$$dx^2 + dy^2 + dz^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2, \quad = r^2dt^2 + 2rR\cos\theta\,dt\,dT + R^2dT^2;$$

that is $\qquad r = \sqrt{E}, \quad R = \sqrt{G}, \quad \cos\theta = \dfrac{F}{\sqrt{EG}},$

and therefore also $\quad \sin\theta = \dfrac{\sqrt{EG - F^2}}{\sqrt{EG}}, \quad = \dfrac{V}{\sqrt{EG}}.$

Writing p, q in place of t, T, and for r, R substituting their values, then, with the foregoing values of u, v, w, U, V, W, Codazzi's six equations are

$$\frac{du}{dq} = \frac{dU}{dp}\cos\theta + \frac{dW}{dp}\sin\theta + w\left(V - \frac{d\theta}{dq}\right) + (U\sin\theta - W\cos\theta)\left(v - \frac{d\theta}{dp}\right),$$

$$\frac{dU}{dp} = \frac{du}{dq}\cos\theta + \frac{dW}{dq}\sin\theta + W\left(v - \frac{d\theta}{dp}\right) + (u\sin\theta - w\cos\theta)\left(V - \frac{d\theta}{dq}\right),$$

$$\left(\frac{dv}{dq} + \frac{dV}{dp}\right)\frac{d^2\theta}{dp\,dq} + \sin\theta\,(uU - wW) - \cos\theta\,(uW + wU) = 0,$$

$$\sqrt{G}\,(u\cos\theta + w\sin\theta) = \sqrt{E}\,(U\cos\theta + W\sin\theta),$$

$$\sqrt{G}\sin\theta\left(v - \frac{d\theta}{dp}\right) + \frac{d\sqrt{G}}{dp}\cos\theta = \frac{d\sqrt{E}}{dq},$$

$$\sqrt{E}\sin\theta\left(V - \frac{d\theta}{dq}\right) + \frac{d\sqrt{E}}{dq}\cos\theta = \frac{d\sqrt{G}}{dp}.$$

[redacted]

I take the opportunity of remarking that Gauss, in § 11 of his memoir, gives the first and second of the formulæ

$$aV^2+a(\varpi F-\omega G)+a'(\omega F-\varpi E)-AE'=0,$$

$$a'V^2+a(\varpi' F-\omega' G)+a'(\omega' F-\varpi' E)-AF'=0,$$

$$a''V^2+a(\varpi'' F-\omega'' G)+a'(\omega'' F-\varpi'' E)-AG'=0$$

(each of them one out of a system of three like equations), where, as before, $$V^2=EG-F^2,$$

$$E',\ F',\ G'=A\alpha+B\beta+C\gamma,\ A\alpha'+B\beta'+C\gamma',\ A\alpha''+B\beta''+C\gamma'',$$

$$\omega,\ \omega',\ \omega''=a\alpha+b\beta+c\gamma,\ a\alpha'+b\beta'+c\gamma',\ a\alpha''+b\beta''+c\gamma'',$$

$$=\tfrac{1}{2}E_1,\ \tfrac{1}{2}E_2,\ F_2-\tfrac{1}{2}G_1,$$

$$\varpi,\ \varpi',\ \varpi''=a'\alpha+b'\beta+c'\gamma,\ a'\alpha'+b'\beta'+c'\gamma',\ a'\alpha''+b'\beta''+c'\gamma'$$

$$=F_1-\tfrac{1}{2}E_2,\ \tfrac{1}{2}G_1,\ \tfrac{1}{2}G_2.$$

These are, in fact, the formulæ (IV),

$$\frac{d^2x}{dp^2}-\left\{\begin{matrix}11\\1\end{matrix}\right\}\frac{dx}{dp}-\left\{\begin{matrix}11\\2\end{matrix}\right\}\frac{dx}{dq}+c_{11}X=0,$$

$$\frac{d^2x}{dp\,dq}-\left\{\begin{matrix}12\\1\end{matrix}\right\}\frac{dx}{dp}-\left\{\begin{matrix}12\\2\end{matrix}\right\}\frac{dx}{dq}+c_{12}X=0,$$

$$\frac{d^2x}{dq^2}-\left\{\begin{matrix}22\\1\end{matrix}\right\}\frac{dx}{dp}-\left\{\begin{matrix}22\\2\end{matrix}\right\}\frac{dx}{dq}+c_{22}X=0,$$

of the memoir, Weingarten "Ueber die Deformation einer biegsamen unausdehnbaren Fläche," *Crelle*, t. c. (1887), pp. 296–310; viz., the symbols correspond to those of Weingarten, as follows:—

$$E,\ F,\ G=a_{11},\ a_{12},\ a_{22},$$

$$V^2=EG-F^2=a=\rho^2,$$

$$E',\ F',\ G'=-c_{11}\sqrt{a},\ -c_{12}\sqrt{a},\ -c_{22}\sqrt{a},$$

$$A,\ B,\ C=\rho X,\ \rho Y,\ \rho Z,$$

$$\omega,\ \omega',\ \omega''=\tfrac{1}{2}\frac{da_{11}}{dp},\quad \tfrac{1}{2}\frac{da_{11}}{dq},\quad \frac{da_{12}}{dq}-\tfrac{1}{2}\frac{da_{22}}{dp},$$

$$\varpi,\ \varpi',\ \varpi''=\frac{da_{12}}{dp}-\tfrac{1}{2}\frac{da_{11}}{dq},\ \tfrac{1}{2}\frac{da_{22}}{dp},\ \tfrac{1}{2}\frac{da_{22}}{dq},$$

and thus Weingarten's symbols, $\left\{ {11 \atop 1} \right\}$, &c., have the values

$$\left\{ {11 \atop 1} \right\} = \frac{1}{a}\left[a_{12}\left(-\frac{da_{12}}{dp}+\tfrac{1}{2}\frac{da_{11}}{dq}\right)+a_{22}\left(\tfrac{1}{2}\frac{da_{11}}{dp}\right)\right],$$

$$\left\{ {12 \atop 1} \right\} = \frac{1}{a}\left[a_{12}\left(-\tfrac{1}{2}\frac{da_{22}}{dp}\right)+a_{22}\left(\tfrac{1}{2}\frac{da_{11}}{dq}\right)\right],$$

$$\left\{ {22 \atop 1} \right\} = \frac{1}{a}\left[a_{12}\left(-\tfrac{1}{2}\frac{da_{22}}{dq}\right)+a_{22}\left(\frac{da_{12}}{dq}-\tfrac{1}{2}\frac{da_{22}}{dp}\right)\right],$$

$$\left\{ {11 \atop 2} \right\} = \frac{1}{a}\left[a_{12}\left(-\tfrac{1}{2}\frac{da_{11}}{dp}\right)+a_{11}\left(\frac{da_{12}}{dp}-\tfrac{1}{2}\frac{da_{11}}{dq}\right)\right],$$

$$\left\{ {12 \atop 2} \right\} = \frac{1}{a}\left[a_{12}\left(-\tfrac{1}{2}\frac{da_{11}}{dq}\right)+a_{11}\left(\tfrac{1}{2}\frac{da_{22}}{dp}\right)\right],$$

$$\left\{ {22 \atop 2} \right\} = \frac{1}{a}\left[a_{12}\left(-\frac{da_{12}}{dq}+\tfrac{1}{2}\frac{da_{22}}{dp}\right)+a_{11}\left(\tfrac{1}{2}\frac{da_{22}}{dq}\right)\right],$$

values which give, as they should do,

$$\left\{ {11 \atop 1} \right\} + \left\{ {12 \atop 2} \right\} = \frac{1}{\sqrt{a}}\frac{d\sqrt{a}}{dp}, \quad \text{and} \quad \left\{ {12 \atop 1} \right\} + \left\{ {22 \atop 2} \right\} = \frac{1}{\sqrt{a}}\frac{d\sqrt{a}}{dq}.$$

The foregoing comparison serves to explain the notation of Weingarten's valuable memoir.

On Complex Primes formed with the Fifth Roots of Unity.

By H. W. Lloyd Tanner. Read March 9th, 1893.

Abstract.

The object of this paper is to explain a method of calculating the complex prime factors of real primes included in the form $10\mu+1$. The only published method which I have met with is due to Kummer. This is not restricted to the particular case here considered; but, as it involves the determination of the G.C.M. of two complex numbers, it is probably more laborious than the method now communicated. The method adopted by Reuschle in the calculation of his Tables does not appear to have been published. The present process is based on the indeterminate equation

$$X^2-5Y^2=4p.$$

A minimum solution of this equation gives the "simplest" prime factor according to Kummer's definition (*Berlin Monatsberichte*, 1870, p. 413), and solutions in which Y is a multiple of 5 give the "primary" prime factors, which Kummer found it necessary to use in order to establish the general law of reciprocity. In solving the equation Lagrange's method turns out to be impracticable, and a short discussion—treated graphically—is introduced, which is sufficient to show the relations between the different solutions. These relations can be expressed in the form

$$\begin{pmatrix} 2, & 0 \\ 0, & 2 \end{pmatrix}\!\!\big(X,\ Y\big) = \begin{pmatrix} a, & 5b \\ b, & a \end{pmatrix}\!\!\big(X',\ Y'\big),$$

and it is interesting to note the intimate connection between these matrices and the complex units. From any solution (X, Y) three numbers A_0, A_1, A_2 are found, A_0 being the integer next greater than $2X/5$; and these serve to determine the values and sequence of the coordinates a_0, a_1, &c., in the required prime factor

$$a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4.$$

The values of a have to satisfy certain conditions, some of which are tested by mere inspection. To give some idea of the facility of the method from the calculator's point of view, it may be stated that the determination of the prime factors of two primes selected at random in the second million (viz., 1562051 and 1671781) was completed in three hours. The only auxiliary table required is a table of squares, and if this extends to the square of 7000 it will suffice for the factorization of all primes in the first nine millions. Tables are appended giving the simplest—and simplest primary—prime factors of all suitable primes less than 10000. The reciprocal factors are also given after the first thousand. For the first thousand the reciprocal factors have already been published, and, instead of giving these again, a comparison is indicated between the factors here given and those published in Reuschle's Tables. The result of the comparison suggests that Reuschle's method of calculation was not the same as that now communicated.

My thanks are again due to my colleague, Mr. Pinkerton, for much and valuable help.

Preliminary Explanations (Arts. 1–4).

1. Let ω be an imaginary fifth root of unity, so that

$$\Omega = 1 + \omega + \omega^2 + \omega^3 + \omega^4$$

is the complex zero. If a_0, a_1, a_2, a_3, a_4 represent any real integers

(coordinates), the expression

$$a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4$$

is a complex integer. This may be indicated by $a\omega$, where a is a functional symbol, so that

$$a\omega^h = a_0 + a_1\omega^h + a_2\omega^{2h} + a_3\omega^{3h} + a_4\omega^{4h}.$$

The norm of $a\omega$,

$$= N.\,a\omega = a\omega\,.\,a\omega^2.\,a\omega^4.\,a\omega^3,$$

is a real integer; and the chief object of the present paper is to determine the coordinates so that this norm may be equal to a given prime p of the form $10\mu+1$.

2. A complex integer, $a\omega$, may be changed in three distinct ways without changing the norm.

(i.) The addition of $j\Omega$, where j is any real integer, positive or negative, increases each of the coordinates by j; but the norm, which is a function of the differences of the coordinates, is not changed.

(ii.) If ω be replaced by ω^h ($h = 1, 2, 3, 4$), the factors in the norm are merely rearranged.

(iii.) The norm is unchanged if $a\omega$ be multiplied by a complex unit; viz., by a complex integer ua such that $N.\,ua = 1$. These units are of three kinds: the real units ± 1; the simple units, ω^k ($k = 0, 1, 2, 3, 4$), and powers of the cyclotomic unit (Kummer's Kreistheilungseinheit) $(\omega+\omega^{-1})^i$, where i is any positive or negative integer. It is known that, for the fifth roots of unity, every unit can be expressed as a product of integral powers of these units. We shall frequently write ϵ or ϵ_1 for $\omega+\omega^{-1}$, and ϵ_2 for $\omega^2+\omega^{-2}$; and

since $$\epsilon_1\,.\,\epsilon_2 = (\omega+\omega^{-1})(\omega^2+\omega^{-2}) = -1,$$

any negative power of ϵ_1 may be replaced by the corresponding positive power of ϵ_2.

3. It appears, then, that all the complex numbers included in the

formula $$\pm\,\epsilon^i.\,\omega^k.\,a\omega^h + j\Omega$$

have the same norm as $a\omega$; and, if the norm is a real prime, the converse is also true. The formula represents a doubly infinite series, for i, j may have any integral values, positive or negative; each term of the series gives 20 arrangements of the coordinates,

since h may have any one of the four values 1, 2, 3, 4, and k is 0, 1, 2, 3, or 4. Finally, the sign of the complex integer may be reversed.

An obvious corollary is that, if the five coordinates of a complex integer are given, without any indication of the sequence, it will, in general, be possible to form 6 complex integers which have different norms. There are, in fact, 120 arrangements of the five coordinates, and these belong in twenties to co-normal integers. Exceptions occur if there are equal coordinates.

4. The discussion which follows enables us to select a particular prime factor of p (the values of h, k, i, j being determinate) as a representative of the whole series, and to determine this representative factor, either directly, by calculating its coordinates, or by derivation from any given prime factor of p.

For this purpose, three auxiliary functions are used—the coordinate-sum, the half-norm, and the reciprocal factors of p.

The coordinate-sum fixes the value of j and the sign. (Art. 5.)

The half-norm fixes the value of i, and furnishes a means of calculating the coordinates a. (Arts. 6, *sqq.*)

The reciprocal factors determine h, k (except that, for certain values of p, the sign of h is left indeterminate), and give further means of lessening the labour. (Arts. 27, *sqq.*)

The Coordinate-Sum (Art. 5).

5. The sum of the coordinates of $a\omega$, namely,

$$a(1) = a_0 + a_1 + a_2 + a_3 + a_4,$$

will be denoted by the letter s. When $a\omega$ is changed to $a\omega + j\Omega$, s becomes $s+5j$. Hence, by properly choosing j, we can make the new $s = 0, \pm 1, \pm 2$. The case $s = 0$ is excluded from consideration; for, if $s = 0$, $a\omega$ is divisible by $1-\omega$, and therefore $Na\omega$ is divisible by $N(1-\omega)$, which is 5. The other cases are reduced to two, viz., $s = 1$, $s = 2$, since we are at liberty to change the sign of $a\omega$.

The coordinate-sum of $e^i . a\omega$ or of $e_2^i . a\omega$, when i is a positive integer, is $2^i . s$, for the coordinate-sum of any product is the product of the coordinate-sums of the factors. By using this it is possible, in an infinite number of ways, to make the new $s = 2$ (or any other selected value except 0 or a multiple of 5). There are advantages in adopting this course, but they are more than counterbalanced by the disadvantage that larger numbers are introduced into the calculations than if we allow the reduced s to be either 1 or 2. It will, therefore, be taken that the coordinate-sum of the standard prime

factor is either 1 or 2. If a standard factor has its coordinate-sum equal to 1, factors with coordinate-sum 2 can at once be found by multiplication into ϵ_1 or ϵ_2.

When $a\omega$ is changed to $a\omega+j\Omega$, s^2 becomes $s^2+10js+25j^2$.

The Half-Norm (Arts. 6–9).

6. The product $a\omega \,.\, a\omega^{-1}$ is a normal product (since ω, ω^{-1} make up a period), and will be called the half-norm.

We have

$$a\omega \,.\, a\omega^{-1} = A_0+A_1(\omega+\omega^{-1})+A_2(\omega^2+\omega^{-2}),$$

where

$$A_0 = a_0^2+a_1^2+a_2^2+a_3^2+a_4^2,$$

$$A_1 = a_0a_1+a_1a_2+a_2a_3+a_3a_4+a_4a_0,$$

$$A_2 = a_0a_2+a_2a_4+a_4a_1+a_1a_3+a_3a_0.$$

The subscripts to the A thus denote the minimum difference, mod. 5, between the subscripts of the factors in the several terms. A_0 is the "square-sum," and A_1, A_2 are the "product-sums" of $a\omega$.

7. A_0 is not altered by a change of ω; A_1, A_2 are interchanged if ω is replaced by ω^2 or ω^3, but are not changed if ω^{-1} is put for ω. Neither of the A is affected when $a\omega$ is changed to $\pm\,\omega^k \,.\, a\omega$.

8. When $a\omega$ becomes $\epsilon^i \,.\, a\omega$, the A change. To determine the new A, we consider the more general case in which $a\omega$ becomes $a\omega \,.\, b\omega$; $b\omega$ being another complex integer whose square-sum and product-sums are B_0, B_1, B_2. Denoting by A_0', A_1', A_2' the corresponding functions for the product $a\omega \,.\, b\omega$, we shall have

$$A_0' = A_0B_0+2A_1B_1+2A_2B_2,$$

$$A_1' = A_0B_1+A_1(B_0+B_2)+A_2(B_1+B_2),$$

$$A_2' = A_0B_2+A_1(B_1+B_2)+A_2(B_0+B_1).$$

It will suffice to prove the second of these. A similar process will serve to establish the others.

We have

$$a\omega \,.\, b\omega = \Sigma a_ib_{-i}+\omega\Sigma a_ib_{1-i}+\omega^2\Sigma a_ib_{2-i}+\omega^3\Sigma a_ib_{3-i}+\omega^4\Sigma a_ib_{4-i},$$

where each Σ includes the five terms for which $i=0, 1, 2, 3, 4$. Hence

$$A_1' = \sum_h \left\{\sum_i a_ib_{h-i}\times\sum_j a_jb_{h+1-j}\right\}; \qquad h, i, j = 0, 1, 2, 3, 4.$$

Now, considering any one value i, the values of j are i, $i\pm 1$, $i\pm 2$.

$$\therefore\ A_1' = \sum_h \sum_i \{a_i^2 b_{h-i} b_{h+1-i} + a_i a_{i+1} b_{h-i} b_{h-i} + a_i a_{i-1} b_{h-i} b_{h-i+2}$$
$$+ a_i a_{i+2} b_{h-i} b_{h-i-1} + a_i a_{i-2} b_{h-i} b_{h-i+3}\}$$
$$= \sum_i \{a_i^2 B_1 + a_i a_{i+1} B_0 + a_i a_{i-1} B_2 + a_i a_{i+2} B_1 + a_i a_{i-2} B_2\}$$
$$= A_0 B_1 + A_1 B_0 + A_1 B_2 + A_2 B_1 + A_2 B_2,$$

which is the equation to be proved.

In particular, taking $b\omega$ to be e_1, $= \omega + \omega^4$, we have $B_0 = 2$, $B_1 = 0$, $B_2 = 1$, so that

$$A_0' = 2A_0 \qquad + 2A_2,$$
$$A_1' = \qquad + 3A_1 + A_2,$$
$$A_2' = A_0 + A_1 + 2A_2.$$

9. If $a\omega$ becomes $a\omega + j\Omega$, the increments of A_0, A_1, A_2 are equal, viz., each is increased by $2sj + 5j^2$. The increment of s^2 (Art. 5) is five times as large. Hence the expressions

$$5A_0 - s^2,\quad A_2 - A_1,\quad A_0 - A_1,\quad A_0 - A_2,\quad A_0 + 2A_1 + 2A_2 - s^2$$

are independent of j. The last is, in fact, zero, for

$$s^2 = A_0 + 2A_1 + 2A_2.$$

The first two are fundamental in the theory, and are denoted by $2X$, Y, namely,

$$2X = 5A_0 - s^2, \quad = 4A_0 - 2A_1 - 2A_2,$$
$$Y = A_2 - A_1.$$

From which we obtain

$$10A_0 = 4X \qquad + 2s^2,$$
$$10A_1 = -X - 5Y + 2s^2,$$
$$10A_2 = -X + 5Y + 2s^2.$$

Concerning X and Y (Arts. 10–24).

10. We have, hence,

$$A_0 - A_1 = \tfrac{1}{2}(X + Y),$$
$$A_0 - A_2 = \tfrac{1}{2}(X - Y).$$

Now,

$$p = N.a\omega = \{A_0+A_1(\omega+\omega^{-1})+A_2(\omega^2+\omega^{-2})\}$$
$$\times\{A_0+A_1(\omega^2+\omega^{-2})+A_2(\omega+\omega^{-1})\}$$
$$= (A_0-A_1)(A_0-A_2)-(A_1-A_2)^2$$
$$= \tfrac{1}{4}(X+Y)(X-Y)-Y^2;$$
$$\therefore\ 4p = X^2-5Y^2.$$

11. The quantity X is essentially positive, for $2X$ is the sum of ten squares, viz.,

$$2X = \Sigma(a_i-a_j)^2, \qquad i = 0, 1, 2, 3, \qquad j = i+1, \ldots, 4,$$

as is at once verified by substituting for the A, in the definition of $2X$, their values given in Art. 6.

It appears, from the definition of Y, that it may be either positive or negative; in fact, the sign of Y is changed when $a\omega$ is changed to $a\omega^2$ or $a\omega^3$. The only other change of $a\omega$ that affects Y, and the only change that affects X, is multiplication by a cyclotomic unit. This we proceed to discuss.

12. Suppose that, when $a\omega$ is replaced by $a\omega\,.\,b\omega$, X, Y become X', Y'. Then we have, using the notation and results of Art. 8,

$$2X' = 4A_0'-2A_1'-2A_2'$$
$$= 2A_0(2B_0-B_1-B_2)-2A_1(B_0-3B_1+2B_2)$$
$$-2A_2(B_0+2B_1-3B_2)$$
$$= (2B_0-B_1-B_2)X+5(B_2-B_1)Y;$$
$$2Y' = 2A_2'-2A_1'$$
$$= 2A_0(B_2-B_1)+2A_1(B_1-B_0)+2A_2(B_2-B_0)$$
$$= (B_2-B_1)X+(2B_0-B_1-B_2)Y.$$

These results are conveniently combined in the matrical equation

$$\begin{pmatrix}2, & 0\\ 0, & 2\end{pmatrix}\begin{pmatrix}X', & Y'\end{pmatrix} = \begin{pmatrix}2B_0-B_1-B_2, & 5(B_2-B_1)\\ B_2-B_1, & 2B_0-B_1-B_2\end{pmatrix}\begin{pmatrix}X, & Y\end{pmatrix}.$$

The terms of the matrix on the right (if we disregard the coefficient 5) are related to $b\omega$ in the same way as X, Y are related to $a\omega$. In particular, they are changed when $b\omega$ is multiplied by ϵ^i. The sign of B_2-B_1 is reversed when $b\omega$ is changed to $b\omega^2$ or $b\omega^3$; but they are not affected by any other change of $b\omega$ which leaves $Nb\omega$ the same.

13. When $b\omega = \epsilon = \omega + \omega^{-1}$, $B_0 = 2$, $B_1 = 0$, $B_2 = 1$. Thus

$$\begin{pmatrix} 2, & 0 \\ 0, & 2 \end{pmatrix}\begin{pmatrix} X_1, & Y_1 \end{pmatrix} = \begin{pmatrix} 3, & 5 \\ 1, & 3 \end{pmatrix}\begin{pmatrix} X, & Y \end{pmatrix}.$$

When $b\omega = \epsilon_2$, we find, in like manner (or by using the remark at the end of the preceding article), that

$$\begin{pmatrix} 2, & 0 \\ 0, & 2 \end{pmatrix}\begin{pmatrix} X_{-1}, & Y_{-1} \end{pmatrix} = \begin{pmatrix} 3, & -5 \\ -1, & 3 \end{pmatrix}\begin{pmatrix} X, & Y \end{pmatrix}.$$

In general, if $a\omega$ be changed to $\epsilon^i . a\omega$, and (X, Y) become X_i, Y_i, we have

$$(X_i, Y_i) = M^i(X, Y),$$

where
$$M = \begin{pmatrix} 3/2, & 5/2 \\ 1/2, & 3/2 \end{pmatrix}.$$

14. If X, Y satisfy the equation

$$X^2 - 5Y^2 = 4p,$$

then X_i, Y_i also satisfy the equation. For, as in Art. 10,

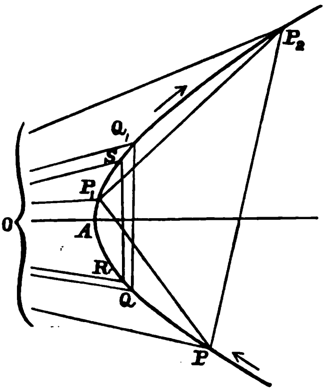
$$X_i^2 - 5Y_i^2 = 4N(\epsilon^i a\omega) = 4Na\omega = 4p.$$

Moreover, if X, Y are integers, so likewise are X_i, Y_i. For, from the equation, it follows that X, Y, if integral, are both even or both odd. Therefore $X_{\pm 1}$, $Y_{\pm 1}$, which are equal to $\frac{1}{2}(3X \pm 5Y)$, $\frac{1}{2}(\pm X + 3Y)$, are integers. It follows, by the method called induction, that X_i, Y_i are integral, where i is any positive or negative integer.

15. The discussion will be simplified by considering X_i, Y_i as the coordinates of a point P_i upon the hyperbola

$$X^2 - 5Y^2 = 4p,$$

or, rather, since X_i is positive, upon the positive branch of this



hyperbola. It is convenient to speak of the system of points P_i [namely, the points whose coordinates are $M^i(X, Y)$ where i is integral] as the range P_i. Two ranges, P_i, Q_i, which have a point in common, are clearly identical. It will now be shown that, when two ranges are not identical, the points of each are separated by the points of the other; or, what is the same thing, there cannot be two points of one range between two consecutive points of the other.

To prove this we note that, O being the origin of coordinates, the area of the triangle OPP_1

$$\begin{aligned} &= \tfrac{1}{2}(XY_1 - X_1Y) \\ &= \tfrac{1}{4}\{X(X+3Y) - (3X+5Y)Y\} \\ &= \tfrac{1}{4}(X^2 - 5Y^2) \\ &= p. \end{aligned}$$

Since this area is positive, it follows that the displacement from P to P_1 along the curve is always in the same sense (indicated by an arrow-head in the figure). Also

$$\Delta OPP_2 > \Delta OPP_1 + \Delta OP_1P_2 > 2p,$$

and, generally, $\Delta OPP_i > ip.$*

On the other hand, if R, S be points between two consecutive points Q, Q_1 of a range, the triangle ORS (taken so as to give a positive area) is included in the triangle OQQ_1, and its area is less than p. Hence R, S cannot belong to one range; we cannot have, for example, two points of the range P_i between two consecutive points of the range Q_i, nor two points of the range Q_i between two consecutive points of the range P_i; in other words, the points of any range P_i are separated by those of any other range Q_i.

It follows that an arc terminated by two consecutive points—say Q, Q_1—of any range contains just one point of every range on the curve.†

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* In this inequality, the integer i is positive; but this merely fixes the sequence of the points P, P_i.

† A well-known example of the dynamics of a particle illustrates this result so aptly that I venture to quote it. It is easy to verify that the area of the sector OPP_1, like that of the triangle, is independent of the position of P. The curve, therefore, is the orbit of a particle moving under repulsion from O varying directly as the distance; and the time of passing from any point to the consecutive point in the same range is constant. Take this as the unit of time. Then a point Q belongs to a range P_i or not, according as the time from P to Q is or is not an integral number of units. If the particle reaches Q t units of time after passing P, it will reach Q_1 $t+1$ units of time from the moment of passing P. If, then, t is integral, Q, Q_1 belong to the range P_i; but, if t is not integral, then, as there is just one integer between t and $t+1$, there is just one point of the range P_i between Q and Q_1.

16. Now in Art. 14 it was shown that, if a range contains an integer-point—that is to say, a point the coordinates of which are integral—then all the points on the range are integer-points. Art. 15 shows that, to detect all the integer-ranges on the curve, it is sufficient to determine the integer-points on an arc QQ_1. The particular arc which is suggested by considerations of symmetry and of ease in calculation, is that shown in the figure, namely, the arc which is bisected by the vertex. In this case, the coordinates of Q, Q_{-1} are $\sqrt{5p}$, $\pm\sqrt{p/5}$. The solutions within these limits—say "reduced" solutions—may be recognized by the value of X being between $2\sqrt{p}$ and $\sqrt{5p}$; or, more conveniently, by the condition that both the expressions $X\pm 5Y$ are positive, while, for every unreduced solution, one is positive and the other negative. From the formulae at the end of Art. 9 it will be seen that, for a reduced solution with $s=1$ or 2, neither of the product-sums A_1, A_2 is positive; for all other solutions, one of the product-sums is positive.

17. It is easy to see the operations by which, from an unreduced solution—or from a prime factor of p, $a\omega$, which gives an unreduced solution—we can form the reduced solution and the corresponding prime factor. If the Y of the given solution is positive (so that the positive product-sum is A_2), the integer point P must be moved backwards to bring it nearer to the vertex. Hence we must act upon (X, Y) with the matrix M^{-1}, repeated if necessary; and the given prime factor $a\omega$ must be multiplied by $\epsilon_2\,(=\omega^2+\omega^{-2})$ to the same extent. When the Y is negative (so that A_1 is the positive product-sum), the matrix M and the unit $\epsilon\,(=\omega+\omega^{-1})$ are to be used, because the point P has to move forwards to approach the vertex.

In the *Berlin Monatsberichte* for June, 1870, Kummer has given, as a criterion for the simplest form of complex numbers, that X must be a minimum. The "reduced" solutions, therefore, give the simplest prime factors of p, and thus become of considerable importance.

18. In general, it happens that a reduced solution is the first that presents itself. The following is an example of how other cases arise. $\omega^2+2\omega^4$ is given by Reuschle as a "simple" prime factor of 11. Bringing this to the proper coordinate-sum (Art. 5), it becomes $1+\omega+\omega^3-\omega^4$, which gives $A_0=4$, $A_1=-1$, $A_2=1$; so that $X=8$, $Y=2$. Now, 8, 2 satisfy the equation

$$X^2-5Y^2=4p=44,$$

but they are not "reduced," because $8-5\times 2$ is negative, and the prime factor is not a "simplest" prime factor. Taking the Y to be positive, we form the solution

$$X_{-1},\ Y_{-1} = M^{-1}(X,\ Y) = (7,\ -1),$$

which is reduced. The prime factor is replaced by

$$(\omega^2+\omega^{-2})(1+\omega+\omega^3-\omega^4) = 1+2\omega^3+\omega^4;$$

and, reducing this to its proper coordinate-sum, by subtracting it from Ω, it becomes

$$\omega+\omega^2-\omega^3,$$

which is one of the "simplest" prime factors of p.*

19. The outcome of the preceding Articles (5–18) is that, from the general form

$$\pm\, \epsilon^i . \omega^k . a\omega^h + j\Omega$$

of the complex integers whose norm is p, we have selected one in which the ambiguous signs and i, j are fixed. Before completing the specification of the standard form, it will be convenient to recall Kummer's definition of a primary number, and then indicate how, from a simplest prime factor of p, a primary prime factor can be determined.

Primary Prime Factors (Art. 20).

20. A primary number $a\omega$ is defined by Kummer as one that satisfies the two congruences

$$a\omega . a\omega^{-1} \equiv a(1) . a(1),\ \bmod \lambda,$$

$$a\omega \equiv a(1),\ \bmod (1-\omega)^2.$$

* It may be remarked that the simplest prime factors are not necessarily the easiest to work with. In particular, if three of the coordinates of $a\omega$ are zero, multiplications become much less laborious than if a simpler form be used in which this is not the case. The real primes which have binomial prime factors are of the form $(a^5+b^5)/(a+b)$, where b is positive or negative, and the prime factors are of the form $a\omega^l + b\omega^m$. Of the 305 primes less than 10,000, only 23 are of this form. Six binomial factors occur as simplest or primary factors in the appended tables—*e.g.*, $p = 31$, 461. Twelve more occur in the tables with three equal coordinates, so that the binomial form is obtained directly by subtracting a proper multiple of Ω. For instance, $p = 61$, 421. The five other primes with binomial factors are here given. The relation of the binomial factor to the simplest factor ($a\omega$) of the tables is indicated in the last line, where ϵ_2 means $\omega^2+\omega^{-2}$.

$$\begin{array}{lccccc} p = & 11, & 181, & 1621, & 4621, & 9931; \\ \left.\begin{array}{c}\text{binomial}\\ \text{factor}\end{array}\right\} = & \omega+2\omega^2, & 4\omega+3\omega^2, & 7\omega+4\omega^2, & 9\omega^2+4\omega^3, & 6\omega^2+11\omega^3, \\ & -\epsilon_2 a\omega+\Omega, & \epsilon_2 a\omega+\Omega, & -\epsilon_2 a\omega+3\Omega, & -\epsilon_2 a\omega+3\Omega, & \epsilon_2 a\omega+3\Omega. \end{array}$$

In the present case, when we are using 5th roots of unity, λ is 5. The first congruence in the notation of Art. 6 becomes

$$A_1(\omega+\omega^4-2)+A_2(\omega^2+\omega^3-2)\equiv 0,\text{*}$$

which may be written

$$A_1(\omega+\omega^4+\omega^2+\omega^3-4)+(A_2-A_1)(\omega^2+\omega^3-2)\equiv 0.$$

Now $$\omega+\omega^4+\omega^2+\omega^3-4=-5,$$

and $\omega^2+\omega^3-2$ is not a multiple of 5. Hence Kummer's first condition is equivalent to

$$A_2-A_1\equiv 0, \text{ mod } 5,$$

that is to say $$Y\equiv 0, \text{ mod } 5.$$

The second congruence merely fixes the absolute term in $a\omega$, and is in nowise peculiar to primary numbers. It will be considered later (Art. 25). In the meantime we proceed to investigate the residues of X, Y, when the modulus is a power of 5, and when the modulus is 4.

Residues of X, Y, for various Moduli (Arts. 21–24).

21. From the equation

$$X^2-5Y^2=4p,$$

we have at once, since $$p=10\mu+1,$$

$$X^2\equiv 4,\text{*}$$

whence $$X\equiv\pm 2.$$

But, for any two consecutive X, we have

$$2X_{i+1}=3X_i+5Y_i,$$

so that $$X_{i+1}\equiv -X_i.$$

Hence the residues of X_i are 2 and -2 alternately.

22. The residues of Y_i are determined by the equation

$$2Y_{i+1}=X_i+3Y_i.$$

On multiplying both sides by 3, and transposing, it is found that

$$Y_{i+1}+Y_i\equiv 3X_i\equiv\pm 1.$$

* Congruences to modulus 5 will occur so frequently in the sequel that the modulus will usually not be expressed. Whenever, as in the present case, the modulus is not stated, it must be understood to be 5.

The consequences of this relation are most simply obtained by using it to write out a cycle of the residues of X_i, Y_i, beginning with an arbitrary pair (2, 0). For a reason that will presently appear, a line is added containing the residues of $3X_i Y_i$,

$$\begin{array}{r|c} X_i \equiv 2 \; -2 \quad 2 \; -2 \quad 2 \; -2 \quad 2 \; -2 \quad 2 \; -2 & 2 \\ Y_i \equiv 0 \quad 1 \quad 3 \quad 3 \quad 1 \quad 0 \quad 4 \quad 2 \quad 2 \quad 4 & 0 \\ 3X_i Y_i \equiv 0 \; -1 \; -2 \quad 2 \quad 1 \quad 0 \; -1 \; -2 \quad 2 \quad 1 & 0 \end{array}$$

The cycle is complete, for every residue of Y occurs once with $X \equiv 2$, and once with $X \equiv -2$. The cycle duly begins again after the tenth column, as will be seen. Hence in any five consecutive integer-points on one range there is one and only one whose ordinate is a multiple of 5. Such a point may be called a primary point, because the prime factors which correspond to it are primary prime factors. The table then shows that $3XY$ measures the distance of (X, Y) from a primary point; and, if we replace $3XY$ by its least residue, positive or negative, we get the distance of the nearest primary point. In this way, if we know a simplest prime factor of p, say $a\omega$, we can determine a simplest primary prime factor of p; namely, $\epsilon^i . a\omega$, where i is the smallest integer, positive or negative, satisfying the congruence $3XY \equiv i$.

23. Similar properties obtain when the modulus is any power of 5. In any set of 5^m consecutive integer-points on the same range there is one and only one which has for ordinate a multiple of 5^m. Between two of these special points there are four whose ordinates are

$$\equiv (1, 3, 3, 1) \times 5^{m-1}, \text{ mod } 5^m,$$

or

$$\equiv (4, 2, 2, 4) \times 5^{m-1}, \text{ mod } 5^m.$$

If (X, Y) be any point whose ordinate is a multiple of 5^{m-1}, then $M^{3XY}(X, Y)$ is a point whose ordinate is a multiple of 5^m.

These theorems can be established by proving that if they are true for m they must also be true for $m+1$; and, since they are true when $m = 1$, they are true for all positive integral values of m.

Now, M being the matrix of Art. 13, if we write

$$M^i = \begin{pmatrix} a_i/2, & 5b_i/2 \\ b_i/2, & a_i/2 \end{pmatrix},$$

we have

$$\frac{a_i + b_i\sqrt{5}}{2} = \left(\frac{a_1 + b_1\sqrt{5}}{2}\right)^i.$$

Therefore $$\frac{a_{5i}+b_{5i}\sqrt{5}}{2}=\left(\frac{a_i+b_i\sqrt{5}}{2}\right)^5,$$

so that $$16a_{5i}=a_i^5+50a_i^3b_i^2+125\,.\,a_ib_i^4,$$

and $$16b_{5i}=5a_i^4b_i+50a_i^2b_i^3+25b_i^5.$$

From the first of these
$$a_{5i}\equiv a_i^5\equiv a_i,\ \text{mod } 5,$$

so that, when i is a power of 5,
$$a_{5i}\equiv a_i\equiv \dots \equiv a_1\equiv -2,\ \text{mod } 5.$$

From the second $\quad b_{5i}\equiv 5b_i,\ \text{mod } 25b_i.$

Now $$b_1=1.$$

Therefore $$b_5\equiv 5,\ \text{mod } 5^2,$$
$$b_{25}\equiv 25,\ \text{mod } 5^3,$$

and, in general, if i is any power of 5,
$$b_i\equiv i,\ \text{mod } 5i.$$

Assume that there is a Y which is a multiple of 5^m, and write $i=5^m$. We have then
$$2Y_i=b_i\,X+a_i\,Y.$$

By supposition $Y=5^m\,.\,H$, where H is integral; and it has been proved that $a_i\equiv -2$, mod 5, $b_i=5^m\beta$, where $\beta\equiv 1$, mod 5. Hence Y_i is divisible by 5^m, $=5^m\,.\,H_i$ say. The last equation becomes, on removing the common factor 5^m,
$$2H_i\equiv X-2H,\ \text{mod } 5\,;$$

or, what is equivalent,
$$H_i+H\equiv 3X,\ \text{mod } 5.$$

This congruence is the same as that of Art. 21, and by similar treatment we learn that one of the points P, P_i, P_{2i}, P_{3i}, P_{4i} has an ordinate which is a multiple of $5i$ $(=5^{m+1})$, so that, as stated, if the theorems enunciated are true for any value of m they are true for $m+1$.

It follows that, from a reduced solution, or indeed any solution, of
$$X^2-5Y^2=4p,$$

we can derive the solutions of
$$X^2-5^{2m+1}\,Y^2=4p,$$

and so find the corresponding prime factors of p.

24. To determine the residues of X_i, Y_i, mod 4, consider the equations

$$\begin{pmatrix} 2, & 0 \\ 0, & 2 \end{pmatrix}(X_3, Y_3) = \begin{pmatrix} 18, & 40 \\ 8, & 18 \end{pmatrix}(X, Y),$$

which give

$$X_3 = 9X + 20Y \equiv X, \text{ mod } 4,$$

$$Y_3 = 4X + \ \ 9Y \equiv Y, \text{ mod } 4.$$

Now, if X, Y are both even, one of them is and the other is not divisible by 4. For, if both were divisible by 4, $X^2 - 5Y^2$ would be a multiple of 16, and if both were $\equiv 2$, mod 4, $X^2 - 5Y^2$ would be a multiple of 8. Hence, if X, Y are even,

$$2X_1, = 3X + 5Y, \equiv 2, \text{ mod } 4,$$

$$2Y_1, \equiv \ \ X + 3Y, \equiv 2, \text{ mod } 4,$$

so that X_1, Y_1 are both odd. It follows that, if P is an "even" point, P_1 and P_2 are odd points (because if P_2 were an even point, P_3 would be odd, in contradiction with what was proved above). It is also seen that, if a point P_i have odd coordinates, one and only one of the points adjacent to P_i has even coordinates, viz., P_{i+1} if $X_i \equiv Y_i$, mod 4, and P_{i-1} if $X_i + Y_i \equiv 0$, mod 4.

Hence, referring to the figure of Art. 15, there will be one and only one even point on the arc terminated by Q_{-1}, Q_3 whose coordinates are $2\sqrt{5p}$, $\pm 4\sqrt{p/5}$. This gives some idea of the comparative facility of solution of the two equations

$$X^2 - 5Y^2 = 4p, \quad \xi^2 - 5\eta^2 = p.$$

To solve the latter, we have only to suppose

$$X = 2\xi, \quad Y = 2\eta.$$

The values of ξ extend from $\sqrt{p}$ to $\sqrt{5p}$ for a minimum solution; about five times the extent that has to be examined for X.

Fixing the Absolute Term (Arts. 25, 26).

25. Kummer's second condition for a primary number is

$$a\omega \equiv a(1), \text{ mod } (1-\omega)^2,$$

and this is applicable to all prime factors, not only those which are primary. For instance, Reuschle, in his *Tafeln*, expresses all the prime factors so as to satisfy this congruence. It is obvious that $1-\omega$ is a factor of

$$a\omega - a(1),$$

and $(1-\omega)^2$ will be a factor if

$$a'.\,\omega \equiv 0,$$

when $\omega = 1$, and $a'\omega$ is the derived function of $a\omega$.

Expressing that the complex number

$$\omega^k\,.\,a\omega$$

satisfies the condition, we obtain

$$a_0 k + a_1(k+1) + a_2(k+2) + a_3(k+3) + a_4(k+4) \equiv 0\,;$$

therefore $\qquad k\,.\,s + a_1 + 2a_2 + 3a_3 + 4a_4 \equiv 0\,;$

therefore $\qquad k \equiv -\,s^3\,(a_1 + 2a_2 + 3a_3 + 4a_4),$

since s is not a multiple of 5.

Thus k is uniquely determined, and the absolute term is fixed.

A change of ω to ω^h does not affect the condition. For, if $a\omega$ satisfies the condition, so that

$$a_1 + 2a_2 + 3a_3 + 4a_4 \equiv 0,$$

then, for $a\omega^h$, the condition becomes

$$h\,(a_1 + 2a_2 + 3a_3 + 4a_4) \equiv 0,$$

which must be satisfied if the former is so.

26. The specification of the standard form of

$$\pm\, e^i\, \omega^k a\omega^h + j\Omega$$

is now substantially complete. The determination of h is a matter of subordinate interest, for it only means selecting one of the four factors of the norm as a representative; and when any one factor

$$a_0 + a_1\omega + a_2\omega^2 + a_4\omega^4 + a_3\omega^3$$

is given, the others are written down by a cyclic transposition of the last four coordinates arranged as above with their subscripts in geometric progression.

Nevertheless it seems worth while to indicate another way of selecting h, k; viz., so that the reciprocal factor of p in its standard form may be obtained from the prime factor with a minimum of correction. Kummer's condition is satisfied by adopting this course, which has the advantage of determining the residues, mod 5, of the coordinates of the prime factor, and so saving some labour in the calculation of the prime factors. At the same time attention is drawn to a certain number—denoted by Q—which is connected with p, and promises to be of some importance in the theory.

Reciprocal factors of p. Residues of coordinates of $a\omega$, mod. 5 (Arts. 27–36).

27. A reciprocal factor, $\psi\omega$, of p is a complex integer such that

$$\psi\omega \,.\, \psi\omega^{-1} = p.$$

The earliest publication relative to these factors appears to be the notice in Legendre's *Théorie des Nombres*, 3rd ed., 1830 (Arts. 549–561). The factors are there symbolized by A, A', A'', A''' (Art. 550), the reciprocal property is verified in Art. 553, and in Art. 558 it is pointed out that the coordinate sum is -1. They appear also in Jacobi's memoir "Ueber die Kreistheilung," 1837 (*Gesammelte Werke*, Bd. VI., pp. 254 &c.), and the mode of calculation (p. 259) fixes the absolute term. The properties of the numbers so calculated suggested to Kummer the definition of primary numbers (*Berlin Monatsberichte*, May, 1850, p. 157). The reciprocal factors for all primes of the form $10\mu+1$, less than 1000, are given in a paper on the quinquisection of the binomial equation (*Proc. Lond. Math. Soc.*, Vol. XVIII., p. 229, 1887). In this table the conditions imposed by Legendre and Jacobi are both satisfied, and the special form of the reciprocal factor is denoted by

$$q\omega, \;= q_0+q_1\omega+q_2\omega^2+q_3\omega^3+q_4\omega^4.$$

It is seen that there are two very distinct sets of primes p. In one set, containing from 75 to 80 per cent. of the whole, it is found that

$$q_0 \equiv \mu-1,\; q_i \equiv \mu+i, \quad i = 1, 2, 3, 4.$$

The primes p which have reciprocal factors of this kind will provisionally be called *perissads*. The rest of the primes, which will be called *artiads*, have reciprocal factors in which

$$q_0 \equiv \mu-1, \quad q_i \equiv \mu, \qquad i = 1, 2, 3, 4.$$

28. Now the product $a\omega \,.\, a\omega^2$ is a reciprocal factor of p, for

$$a\omega \,.\, a\omega^2 \times a\omega^{-1} \,.\, a\omega^{-2} = N \,.\, a\omega = p.$$

But the coordinate sum of $a\omega \,.\, a\omega^2$ is s^2; that is to say, either 4 or 1. In the former case the coordinate sum can be reduced to the proper value, -1, by subtracting Ω; in the latter case, by changing the sign of the product. The immediate problem then becomes the determination of $a\omega$, so that

$$a\omega \,.\, a\omega^2 = q\omega + \Omega, \quad \text{when } s = 2,$$

or else

$$a\omega \,.\, a\omega^2 = -q\omega, \quad \text{when } s = 1.$$

29. In analysing these sys
equation of each system (viz.,
form. The others are replace
difference between the left sides
the difference between the c
differences take a characteristic f
$s = 2$, taking $i = 2, 4$,

$$q_1 - q_2 = \{a_2^2 + (a_1 + a_3)(a_0 + a_4)\} -$$

$$= \{a_2^2 + (a_1 + a_3)\, a_4 + a_3 a_0 + a_1$$

and this, on interchanging the seco
cancelling the common term $a_1 a_0$, b

$$= \{a_2^2 - (a_3 + a_0)\, a_2 + a_3 a_0\} -$$

$$= (a_2 - a_3)(a_2 - a_0) - (a_4 - a_1$$

similarly, $q_3 - q_4 = (a_1 - a_4)(a_1 - a_2$

30. Hence

$$q_1 - q_2 - q_3 + q_4 = (a_2 - a_3)(a_2 -$$

$$-(a_1 - a_4)(a_1 -$$

$$= (a_3 - a_2)(-a_2$$

$$-(a_4 - a$$

31. Now, from the known residues of q_i (Art. 27), it is seen that

$$q_1 - q_2 - q_3 + q_4 \equiv 0.$$

Hence $\qquad (a_3 - a_2)^2 + (a_3 - a_2)(a_4 - a_1) - (a_4 - a_1)^2 \equiv 0;$

therefore $\qquad \{a_3 - a_2 - 2(a_4 - a_1)\}^2 \equiv 0.$

Hence, if we write $\qquad a_4 = a_1 + \delta,$

we have also $\qquad a_3 \equiv a_2 + 2\delta.$

Using these values for a_3, a_4, we find

$$\begin{aligned} s &\equiv a_0 + a_1 + a_2 + a_2 + 2\delta + a_1 + \delta \\ &\equiv a_0 + 2a_1 + 2a_2 + 3\delta, \end{aligned}$$

a congruence which may also be written in the equivalent forms

$$a_0 + 2a_1 + 2a_2 \equiv s + 2\delta,$$

$$-2a_0 + a_1 + a_2 \equiv 3s + \delta.$$

32. Again, letting the upper and lower signs correspond to the two cases $s = 2, 1$, we find

$$\begin{aligned} \pm (q_0 - q_1) &= (a_0 - a_3)(a_0 - a_1) - (a_2 - a_1)(a_2 - a_4) \\ &\equiv (a_0 - a_2 - 2\delta)(a_0 - a_1) - (a_2 - a_1)(a_2 - a_1 - \delta) \\ &\equiv -(2a_0 - a_1 - a_2)^2 - (2a_0 - a_1 - a_2)\,\delta \\ &\equiv -(3s + \delta)\,3s \\ &\equiv s^2 + 2s\delta. \end{aligned}$$

Thus $\qquad q_0 - q_1 \equiv -1 - \delta \qquad$ when $s = 2$,

$\qquad q_0 - q_1 \equiv -1 - 2\delta \qquad$ when $s = 1$.

For an artiad $\qquad q_0 - q_1 \equiv -1 \quad$ (Art. 27),

so that, whether $s = 2$ or 1, $\quad \delta \equiv 0$.

On the other hand, for a perissad, $q_0 - q_1 \equiv -2$, and thus

$$\delta \equiv 1 \text{ when } s = 2, \quad \delta \equiv 3 \text{ when } s = 1,$$

results which may be combined in the formula

$$\delta \equiv 3s.$$

The last congruence of Art. 31 then gives

$a_1 + a_2 \equiv 2a_0 + 3s \quad$ when p is an artiad,

$a_1 + a_2 \equiv 2a_0 + s \quad$ when p is a perissad.

33. The difference a_1-a_2 can be expressed by means of Y. We have, namely,

$$Y = a_0a_2+(a_2a_1+a_2\delta)+(a_1^2+a_1\delta)+(a_1a_3+2a_1\delta)+(a_3a_0+2a_0\delta)$$
$$-a_0a_1-a_1a_2-(a_2^2+2a_2\delta)-(a_3a_1+2a_1\delta+a_2\delta+2\delta^2)-(a_1a_0+a_0\delta),$$

whence, after some reductions,

$$2Y \equiv s(a_1-a_2+2\delta).$$

From which, since s is not a multiple of 5,

$$a_1-a_2 \equiv 2Ys^3 \qquad \text{when } p \text{ is artiad,}$$
$$a_1-a_2 \equiv 2Ys^3-s \qquad \text{when } p \text{ is perissad.}$$

Combining these with previous results, it will be found that, when p is artiad,

$$a_1 \equiv a_0+Ys^3-s \equiv a_4, \qquad a_2 \equiv a_0-Ys^3-s \equiv a_3,$$

and, when p is perissad,

$$a_1 \equiv a_0+Ys^3, \quad a_2 \equiv a_0-Ys^3+s, \quad a_3 \equiv a_0-Ys^3+2s, \quad a_4 \equiv a_0+Ys^3+3s.$$

34. It only remains to determine the residue of a_0, mod 5. For this purpose we use the expression

$$(\psi_0 =)\ a_0^2+(a_1+a_4)(a_2+a_3),$$

which is equal to q_0+1 or $-q_0$, according as s is 2 or 1 (Art. 28).

Observing that
$$a_1+a_4 \equiv 2a_0+2Ys^3-2s,$$
$$a_2+a_3 \equiv 2a_0-2Ys^3-2s,$$

for all values of p, we have

$$\psi_0 \equiv a_0^2-(a_0-s)^2+Y^2s^2$$
$$\equiv 2a_0s-s^2+Y^2s^2,$$

so that
$$a_0 \equiv 3s+2Y^2s+3s^3\,.\,\psi_0.$$

Hence, when $s=2$, so that

$$\psi_0 = q_0+1 \equiv \mu,$$
$$a_0 \equiv 1-Y^2-\mu \equiv -\mu-Y^2+1,$$

and when $s=1$, so that

$$\psi_0 = -q_0 \equiv -\mu+1,$$
$$a_0 \equiv 3+2Y^2+2\mu-2 \equiv 2\mu+2Y^2+1.$$

35. The following table includes a summary of the results, and gives, for a real prime p, the residues of the coordinates expressed in terms of Y and μ, $= (p-1)/10$. The multiple of $\mu+Y^2$ in the third column is a part of the residue of each of the coordinates on the same line,

p	s		a_0	a_1	a_2	a_3	a_4
artiad......	2	$-\mu-Y^2+$;	1,	$-2Y-1$,	$2Y-1$,	$2Y-1$,	$-2Y-1$
perissad...	2	$-\mu-Y^2+$;	1,	$-2Y+1$,	$2Y-2$,	$2Y$,	$-2Y+2$
artiad......	1	$2\mu+2Y^2+$;	1,	Y,	$-Y$,	$-Y$,	Y
perissad...	1	$2\mu+2Y^2+$;	1,	$Y+1$,	$-Y+2$,	$-Y-2$,	$Y-1$

36. For practical application it is convenient to have these worked out for the several residues of Y, as in the table given below.

		p artiad	p perissad	
$s=2$, $X\equiv-2$	$Y\equiv0$	$-\mu+$; 1, 4, 4, 4, 4	$-\mu+$; 1, 1, 3, 0, 2	6
	$Y\equiv1$	$-\mu+$; 0, 1, 0, 0, 1	$-\mu+$; 0, 3, 4, 1, 4	2
	$Y\equiv-1$	$-\mu+$; 0, 0, 1, 1, 0	$-\mu+$; 0, 2, 0, 2, 3	10
	$Y\equiv2$	$-\mu+$; 2, 1, 4, 4, 1	$-\mu+$; 2, 3, 3, 0, 4	8
	$Y\equiv-2$	$-\mu+$; 2, 4, 1, 1, 4	$-\mu+$; 2, 1, 0, 2, 2	4
$s=1$, $X\equiv2$	$Y\equiv0$	$2\mu+$; 1, 0, 0, 0, 0	$2\mu+$; 1, 1, 2, 3, 4	1
	$Y\equiv1$	$2\mu+$; 3, 3, 1, 1, 3	$2\mu+$; 3, 4, 3, 4, 2	5
	$Y\equiv-1$	$2\mu+$; 3, 1, 3, 3, 1	$2\mu+$; 3, 2, 0, 1, 0	7
	$Y\equiv2$	$2\mu+$; 4, 0, 1, 1, 0	$2\mu+$; 4, 1, 3, 4, 4	9
	$Y\equiv-2$	$2\mu+$; 4, 1, 0, 0, 1	$2\mu+$; 4, 2, 2, 3, 0	3

The term in μ is common to all the coordinates in the same line, so that the first line, for instance, shows that in a primary prime factor (with coordinate sum 2) of an artiad p,

$$a_0 \equiv -\mu+1, \quad a_1 \equiv -\mu+4, \equiv a_2 \equiv a_3 \equiv a_4;$$

and for a perissad p the coordinates are

$$a_0 \equiv -\mu+1 \equiv a_1, \quad a_2 \equiv -\mu+3, \quad a_3 \equiv -\mu, \quad a_4 \equiv -\mu+2.$$

The last column indicates the order in which the prime factors occur if, starting with any one, we multiply it by ϵ, ϵ^2, &c. We pass from a prime factor marked i to one marked j by multiplying the former by ϵ^{j-i}, and correcting the coordinate-sum if necessary. This is easily verified directly, or, by observing that $s = 2, 1$ are tantamount to $X \equiv -2, 2$ respectively (for $2X = 5A_0 - s^2$, and therefore $X \equiv 2s^2$), so that we are in effect repeating the table of Art. 22.

Sign of Y and Q (Arts. 37, 38).

37. This will be a convenient place to give some account of the mode of fixing the sign of Y in the tables appended. When p is a perissad, there is no difficulty, for, as will be seen by a reference to Art. 35 or Art. 36, a change of the sign of Y gives a different set of residues for the coordinates, and even for a primary factor, namely when $Y \equiv 0$, the sequence of the residues is altered. Consider, for instance, the perissad 811, for which $X = 57$, $Y = \pm 1$, $\mu \equiv 1$, $s = 1$. The coordinates must be $\equiv 0, 1, 0, 1, 4$ or $0, 4, 2, 3, 2$, according as Y is positive or negative. Now the coordinates are found to be $0, -1, 2, 3, -3$, which agree with the second, and, however they may be arranged, are incompatible with the first. Thus $Y = -1$.

It is different with an artiad. Take, for example, the artiad 211, for which $X = 32$, $Y = \pm 6$, $\mu \equiv 1$, $s = 1$. Now, when Y is positive, the coordinates must be $\equiv 0, 0, 3, 3, 0$; if negative, they are $\equiv 0, 3, 0, 0, 3$. The coordinates are found to be $0, 0, 0, -2, 3$. The prime factor may be written $3\omega^4 - 2\omega^3$ or $-2\omega^2 + 3\omega^3$ consistently with $Y = 6$; it may be written $3\omega - 2\omega^4$ or $-2\omega + 3\omega^4$ for $Y = -6$. The residues, in fact, give no help in fixing the sign of Y.

38. Under these circumstances we make use of a function already mentioned, in Art. 30, viz.,

$$\begin{aligned} Q_s &= q_1 - q_2 - q_3 + q_4 \\ &= (-1)^s \left\{ (a_3 - a_2)^2 + (a_3 - a_2)(a_4 - a_1) - (a_4 - a_1)^2 \right\}.^* \end{aligned}$$

In the first place it will be observed that when ω is changed to ω^2 in the prime factor or the reciprocal factor, the sign of Q is reversed. The same is true of Y (Art. 11). Hence, if we fix the sign of Q, the sign of Y is also fixed.

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* The symbol $(-1)^s$ is used on the assumption that s is 1 or 2. If other values of s have to be considered, $(-1)^s$ must be replaced by the Legendrian symbol $(2s/5)$; for it is 1 when $s \equiv \pm 2$, and -1 if $s \equiv \pm 1$.

Now, making the substitutions (Art. 31)

$$a_4-a_1=\delta,$$

$$a_3-a_2-2\,(a_4-a_1)=5\delta',$$

we have $$Q=(-1)^s\left\{5\delta^2+25\delta\,\delta'+25\delta'\delta'\right\}$$

$$\equiv(-1)^s\,5\delta^2,\ \text{mod } 25,$$

$$\equiv 5,\ \text{mod } 25,\ \text{if } p \text{ is perissad};$$

since, by Art. 32, $\delta\equiv 3$ when $s=1$, and $\delta\equiv 1$ when $s=2$.

When p is artiad it will be found that

$$Q\equiv\pm\,5^{m-1},\ \text{mod } 5^m.$$

The prime factors in the following tables have been arranged so that the upper sign is true, viz.,

$$Q\equiv 5^{m-1},\ \text{mod } 5^m,^*$$

and the sign of Y thus becomes determinate.

Example of first process of calculation (Arts. 39–51).

39. Some account will now be given of the details of the calculation of the complex factors of a given prime, and, to illustrate the method, we take as example $p=1671781$.

40. The first step is to solve the equation

$$X^2-5Y^2=6687124\ (=4p).$$

Since the terminal digits of $5Y^2$ are 80, 000, 500, or 20, for an even Y, and e05, 125, e45 for an odd Y (e being any even digit), it follows that X^2 must end in

04, 124, 624, 44; o29, 249, o69,

where o stands for an odd digit. From Art. 16 it appears that the minimum value of X lies between 2585 ($<2\sqrt{p}$) and 2892 ($>\sqrt{5p}$); but of the 306 integers between these limits only 54 give X^2 with a proper ending, so that at worst 54 trials have to be made. A convenient way of making a trial is to add $-4p$, in the form 7312876 or 6712876 or 6692876, to the tabulated values of X^2. The sum,

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* There are some developments in which a more convenient assumption would be

$$Q\equiv-(-5)^{m-1},\ \text{mod } 5^m;$$

but this question, and others relating to Q, I hope to consider in a separate note.

multiplied by ·2, is an exact square, Y^2, when the proper value of X is reached. In the present case it happens that the first trial succeeds, and the minimum solution is

$$X = 2587, \quad Y = \pm 33.$$

41. Hence, by Art. 9,

$$s = 1, \quad A_0 = 1035, \quad (A_2, A_1) = (-242, -275),$$

and A_0 is now to be expressed (Art. 6) as a sum of five squares

$$1035 = a^2+b^2+c^2+d^2+e^2.$$

In writing down these decompositions the signless roots are arranged in order of magnitude,

$$a \nless b \nless c \nless d \nless e.$$

The greatest value for a is 32, the integral part of $\sqrt{1035}$, and there are 200 decompositions, beginning with 32, 3, 1, 1, 0, and ending with 17, 16, 15, 12, 11. It is, however, needless to write out all these, and I have found it convenient to write down all the decompositions containing one value of a, and to examine these before proceeding to the next value of a. On this plan the decompositions noted did not extend very far beyond the one required; and, on the other hand, writing down a number of them at a time was almost mechanical and less liable to errors of omission.

42. The first test applied is the coordinate-sum. The values of a, b must be such that, on properly fixing the signs,

$$\pm a \pm b \pm c \pm d \pm e = s,$$

where s is known. Hence, if the sum of the terms, all taken positively, be S, the sum of the positive terms of a proper decomposition is $\frac{1}{2}(s+S)$, and that of the negative terms $\frac{1}{2}(s-S)$. If, then, we can express $\frac{1}{2}(S+s)-a$ as a sum of any of the elements b, c, d, e, these, with a, furnish the positive terms of a proper-signed decomposition. Or, again, if $\frac{1}{2}(S-s)-a$ can be similarly partitioned, these elements, with a, have the negative sign in another set of signed terms. If neither partition is possible, the set (a, b, c, d, e) fails.

An important case of failure is when $a-s > 2\sqrt{A_0-a^2}$. For we have

$$2\sqrt{A_0-a^2} = 2\sqrt{(b^2+c^2+d^2+e^2)} > b+c+d+e.$$

Hence $\qquad a-s > b+c+d+e,$

that is to say $\qquad a-b-c-d-e > s.$

From this it is seen that in general the greatest coordinate cannot be greater than $2\sqrt{A_0/5}$. In the numerical example, we find that the greatest value of a, according to this formula, is 28, and, by putting 29 for a in the inequality above, it is verified that 29 is an impossible value. Thus the decompositions containing 32 or 31, or 30 or 29, are at once ruled out.

43. We have now to deal with signed sets of terms, and of course one decomposition into squares may give more than one set of signed terms. The second test is the table of Art. 36. In the example before us,

$$s = 1, \quad 2\mu \equiv 1, \quad Y \equiv \pm 2;$$

and the table shows that the coordinates of the prime factor sought must have the residues

$$0, 1, 2, 2, 1; \quad \text{or } 0, 2, 4, 0, 0; \quad \text{or } 0, 3, 3, 4, 1;$$

and a set of signed terms whose residues do not agree with one of these is not right. For example,

$$\pm 24, \ \mp 21, \ \mp 4, \ \pm 1, \ +1,$$

which gives the proper coordinate-sum when all the upper signs or when all the lower signs are taken, is excluded, for it contains no multiple of 5.

44. If a signed set of terms satisfies both tests, the values of A_1, A_2 are calculated, and if these have the required values, the required prime factor is found. The work is arranged thus

$$\begin{array}{ccccc} a & b & c & d & e \\ \hline & ab & ac & ad & ae \\ & & bc & bd & be \\ & & & cd & ce \\ & & & & de \end{array}$$

where now $a, b, \ldots e$ represent the terms, each with its proper sign. If $a, b, \ldots e$ are in the right order (namely $a = a_0$, $b = a_1$, ... $e = a_4$), then the sum of the four products in the diagonal line, and the product in the right-hand upper corner, is A_1. The sum of the other five products is A_2; but it is unnecessary to calculate A_2 (save for verification) because $A_1 + A_2$, $= (A_0 - s^2)/2$ is the same for all sets which satisfy the first test.

45. In general there is ambiguity about the sequence of $a, b, \dots e$ (as in the instance worked below), and it is desirable to be able to find A_1, A_2 for every arrangement of the terms, without rewriting. This is done by means of the following diagrams, in which the two sets of five products each are denoted by crosses and noughts respectively.

$$
\begin{array}{rrrr} \times & 0 & 0 & \times \\ & \times & 0 & 0 \\ & & \times & 0 \\ & & & \times \end{array}
\qquad
\begin{array}{rrrr} \times & \times & 0 & 0 \\ & 0 & \times & 0 \\ & & 0 & \times \\ & & & \times \end{array}
\qquad
\begin{array}{rrrr} \times & 0 & \times & 0 \\ & \times & 0 & 0 \\ & & 0 & \times \\ & & & \times \end{array}
\qquad
\begin{array}{rrrr} \times & \times & 0 & 0 \\ & 0 & 0 & \times \\ & & \times & 0 \\ & & & \times \end{array}
\qquad
\begin{array}{rrrr} \times & 0 & 0 & \times \\ & 0 & \times & 0 \\ & & \times & \times \\ & & & 0 \end{array}
\qquad
\begin{array}{rrrr} \times & 0 & \times & 0 \\ & 0 & 0 & \times \\ & & \times & \times \\ & & & 0 \end{array}
$$

abcde	*abdec*	*abced*	*abedc*	*abdce*	*abecd*
adbec	*aebcd*	*aebdc*	*adbce*	*acbed*	*acbde*

Beneath each diagram two arrangements of the coordinates *abcde* are shown. Each of these may be read backwards or forwards and may begin at any letter, so that to each diagram there are 20 arrangements (10 for each signature), and, since no arrangement is repeated, there are 120 in all, which is right. For the upper signatures, in which a, b are adjacent, A_1 is the sum of the $\times$, and A_2 of the 0. For the lower signatures, in which a, b are not adjacent, A_1 is the sum of the 0, and A_2 of the $\times$.

46. As an example of the use of these diagrams, take the set of signed terms 23, -21, $+6$, -5, -2, the residues of which are 3, 4, 1, 0, 3, agreeing with the third set named in Art. 43, viz., 0, 3, 3, 4, 1. It is seen that two arrangements of the terms under trial are available. We try

$$
\begin{array}{rrrrr}
-5 & -2 & 23 & -21 & 6 \\
\hline
 & 10 & -115 & 105 & -30 \\
 & & -46 & 42 & -12 \\
 & & & -483 & 138 \\
 & & & & -126
\end{array}
$$

The principal diagram fails to give $(A_1, A_2) = (-242, -275)$. In fact it is visible, without doing the addition, that

$$A_1 \,(= 10-46-483-126-30) \text{ is } < -600;$$

and that A_2 is positive. Either of these results shows that a, b, c, d, e is not right. The other arrangement, consonant with the second

test, is a, c, b, d, e, the lower signature of the last diagram. A_1 (given by the 0 of that diagram) $= -275$; $A_2 = -242$. Hence

$$Y = A_2 - A_1 = 33,$$

and this agrees with the indication, in the table of Art. 36, that $Y \equiv -2$ for the residues 0, 3, 3, 4, 1. We find, then, that the simplest prime factor of 1671781 is

$$-5+23\omega-2\omega^2-21\omega^3+6\omega^4.$$

47. Two other methods of reducing the work may be noted. As soon as the residues of a trial-set are written down, the value of Y is known, and therefore also the values of A_2, A_1 separately. These two values do not end in the same digit, unless the simplest prime factor is primary.* Therefore we can, by working with the final digits only of the products ab, &c., at once exclude several—in the long run one-half—of the failures. Secondly, it is obvious that two or more trial sets, which differ only by their signs, can be tested in one multiplication scheme, by putting two or more rows of signs before the products.

48. In calculating the prime factor written above, it was necessary to write down 103 five-square expressions for A_0, beginning with $a = 28$. Of these 43 were excluded by the first test ($s = 1$), and 17 were not examined, as they came after the solution was obtained. On the other hand, 13 satisfied this test doubly. The "second test" excluded 6 single and 2 double cases; and the values of A_2, A_1 were examined in 24 single and $10\frac{1}{2}$ double cases (the solution being the first of a double case), so that in all 35 multiplication schemes like that in Art. 46 were formed.

49. To obtain the simplest primary factor, we note that, since $X \equiv 2$ and $Y \equiv -2$, $3XY \equiv -2$; and the primary factor is $\epsilon_2^2 . a\omega$, where

$$a\omega = -5+23\omega-2\omega^2-21\omega^3+6\omega^4.$$

Now, multiplying by ϵ_2 means replacing each coordinate (a_i) by the sum of the two opposite coordinates ($a_{i+2}+a_{i-2}$), just as multiplying by ϵ means replacing each coordinate (a_i) by the sum of the two

* When $Y \equiv 0$ there is a simplification in the work, as the "second test" gives only two sets of residues instead of three. The symmetry of this case, and the ease of the computations, suggested this condition as proper for a canonical form of a prime factor, before I found—with no small pleasure—that it was Kummer's first condition for a primary number.

[redacted]

adjacent coordinates $(a_{i+1}+a_{i-1})$. In multiplying by e_2^2, the coordinate-sum becomes $4s$ (that is to say, in the present case, 4), and the coordinate-sum is corrected by subtracting each of the coordinates in the product from unity. The work can be done in three lines, thus:

$$
\begin{array}{rrrrrr}
a\omega = & -\ 5 & 23 & -2 & -21 & 6 \\
e_2 a\omega = & -23 & -15 & 1 & 18 & 21 \\
e_2^2 a\omega = & 19 & 39 & -2 & -38 & -14
\end{array}
$$

and the simplest primary factor $(=\Omega-e_2^2 a\omega)$ is

$$-18-38\omega+3\omega^2+39\omega^3+15\omega^4.$$

50. The product $a\omega \,.\, a\omega^2$, corrected for coordinate-sum, is the reciprocal factor; and $a\omega$ may be replaced by the primary factor or any other prime factor. The coefficient of ω^{2i} in the product is

$$a_i^2+(a_{i+1}+a_{i-1})(a_{i+2}+a_{i-2}),$$

and this suggests a convenient arrangement of the work, which makes it possible indeed to write down the coordinates of the reciprocal factor at once for moderate values of p. Having written down the coordinates of $a\omega^3$, place under each of them (1) its square, (2) the product of the adjacent sum and the opposite sum (where "adjacent sum" means the sum of the two coordinates adjacent to the one treated, and similarly for the "opposite sum"). The sum of the square and product is the coordinate of $a\omega \,.\, a\omega^2$, in its proper place. For example, taking $a\omega$ to be the simplest prime factor obtained in Art. 46, we have

$a\omega^3$,	-5	-2	6	23	-21
square	25	4	36	529	441
product	$-23 \cdot 29$	$1 \cdot 2$	$-21 \cdot 26$	$15 \cdot 7$	$18 \cdot 4$
$a\omega \,.\, a\omega^2 =$	-642	$+6\omega$	$-510\omega^2$	$+634\omega^3$	$+513\omega^4$

Hence the reciprocal factor is

$$q\omega = 642-6\omega+510\omega^2-634\omega^3-513\omega^4,$$

and from this the cyclotomic of the fifth degree can be formed by

the formulæ in the paper (*Proc. Lond. Math. Soc.*, Vol. XVIII.) already cited.*

51. The value of Q is

$$-6-510+634-513 = -395 \equiv 5, \text{ mod } 25,$$

as it should be.

When p is artiad, the sign of Y is undetermined until this stage. The convenient course is to assume one sign for Y, and if this leads to $Q \equiv -5^m$, mod 5^{m+1}, it is only necessary to replace $a\omega$ and $q\omega$ by $a\omega^2$, $q\omega^2$ respectively.

Second process (Arts. 52–54).

52. In the above process there is a considerable amount of labour wasted in making trials which fail, and it is clear that much of the waste work might be utilized for the factorization of other primes. This remark suggests a second mode of attacking the problem. Instead of seeking the complex factors of a given prime, we may seek all the primes that belong to a given A_0. Of the decompositions

$$A_0 = a^2+b^2+c^2+d^2+e^2,$$

we select those which make $s = 1$ if A_0 is odd, or $s = 2$ if A_0 is even. Any set of signed terms which does not satisfy this test can [by adding any (the same) integer to each term and reversing the sign if necessary] be reduced to a set in which the "first test" is satisfied, but with a different A_0. By this exclusion we do not miss any primes, but merely avoid repetitions of the same prime under a different A_0.

For the same reason it is desirable to use only those signed terms which belong to reduced solutions; the excluded cases occur as reduced solutions for a smaller A_0.

53. When a set of signed terms is selected, the multiplication scheme of Art. 44 is formed; and then, by using the diagrams of Art. 45, we compute six pairs of product-sums (A_2, A_1) which reduce however to three pairs when there are two equal terms, to two pairs

* An unchecked calculation gives the following values for the coefficients of the cyclotomic $\eta^5+\eta^4+P_2\eta^3+P_3\eta^2+P_4\eta+P_5$ in this case :—

$$P_2 = -668712,\ P_3 = -215124779,\ P_4 = -28654017881,$$
$$P_5 = 1685452024703.$$

if there are two pairs of equal terms, and to one pair if three of the terms are equal.

It is sufficient to record one product-sum of each pair, the other being given by the equation

$$A_1+A_2 = -\tfrac{1}{2}(A_0-s^2).$$

For this we choose the greater of the pair, and calling it A_2 we shall have Y positive. It will be observed that A_2 is numerically less than A_1, and that the entry corresponds to a reduced or an unreduced solution according as it is $\leqq 0$ or >0. The upper signature of the diagram (Art. 45) is proper when A_2 is the sum of the "noughts," the lower when it is the sum of the crosses. If this signature does not agree with the table of Art. 36, we must interchange A_2, A_1 and replace Y, ω by $-Y$, ω^2.

54. Consider, for example, the scheme of Art. 46, which gives the product-sums

$$158, \quad -51\times, \quad 81\times, \quad 209, \quad -194, \quad -242\times,$$

those marked $\times$ being the sums of the crosses. The first, third and fourth are not reduced, and the sixth has been worked out. The second, taking $A_2 = -51$, gives $A_1 = -466$ and $Y = 415$, $\equiv 0$. Hence, $N.a\omega = (A_0-A_1)(A_0-A_2)-(A_1-A_2)^2 = 1457861$,

for which $2\mu \equiv 2$. The table of Art. 36 gives a choice of residues 3, 2, 2, 2, 2 or 3, 3, 4, 0, 5; and the lower signature of the second diagram (Art. 45), namely, $a\,e\,b\,c\,d$, becomes -5, 6, -2, 23, -21. This agrees with the 3, 3, 4, 0, 5, if we begin with -2, and go forwards. Thus the assumption as to the sign of Y was correct, and the complex factor of 1457861 in the standard form is

$$-2+23\omega-21\omega^2-5\omega^3+6\omega^4,$$

and $$(2587)^2-5\,(415)^2 = 4\times 1457861.$$

Similarly, from the fifth entry, it will be found that 1652341 is the norm of $6-5\omega-2\omega^2-21\omega^3+23\omega^4$ and $Y = 129$.

Conditions that $a\omega$ should be a complex prime (Arts. 55, 56).

55. When a table of primes is at hand it can be determined by inspection whether the norms obtained are or are not primes; in other words, whether $a\omega$ is or is not a complex prime. This can also be determined, without using a table of primes, in the following way.

The real factors of a norm of a complex integer are of three kinds. First we have primes of the form $10\mu+1$ (say $p_1, p_2, \ldots$); next, primes of the form $10\mu-1$ (say $q_1, q_2, \ldots$), which must appear as powers q^{2t}, t being an integer; lastly, there are primes of the form $10\mu \pm 3$ and 2 (say $r_1, r_2 \ldots$), which appear as powers r^{4t}.

Now, if $Na\omega$ contain r as a factor, r will be a factor of all the differences of the coordinates of $a\omega$, and can be immediately detected.

If $Na\omega$ contain a prime, q, as a factor, this prime will be a factor of all the differences of the coordinates of the "reciprocal factor" $a\omega \,.\, a\omega^4$.

The presence of two primes $p_1 \,.\, p_2$, or of a square p^2, as factors of $Na\omega$, can be recognized in two ways, when each of the p is of the form $10\mu+1$.

The first way depends on the fact that there are two integer points in the reduced half-segment AQ_1 (see figure to Art. 15).* If there are more than two p-factors, there will be more than two integer points, but the number depends upon how many of the p are equal. To detect these it will be needful to examine the suitable values of X between $2\sqrt{N}$ and $\sqrt{5N}$, where N represents $Na\omega$.

The second way depends on the fact that, corresponding to any one integer point, there will be two (or more) different sets of coordinates; that is to say, there will be two or more complex numbers which have the same X, Y (and therefore the same A_0, A_1, A_2, and s), and may be taken to have the proper absolute terms, and yet are not included in one formula $a\omega^k$. The existence of such numbers is indicated by the same product-sum occurring in two different multiplication schemes belonging to one square-sum.

56. The former process is the easier when we are dealing with an isolated complex number. For instance, when $Na\omega = 1652341$, there are four integer points on the reduced half-segment, viz., (2587, 129), (2613, 209), (2817, 515), (2847, 547); and, in fact,

$$1652341 = 41 \times 191 \times 211.$$

The other method is better if we are examining all the complex numbers that belong to a given square-sum. If a product-sum occurs only in connexion with one complex number, and the norm of this number is free from q and r factors, then the complex integer is

* This theorem is due to Tchébicheff, and is quoted in Prof. Mathews' admirable text-book on the *Theory of Numbers*, Pt. I., p. 270.

a complex prime and its norm is a real prime. For instance, if $A_0 = 39$, there are five sets of signed terms

$$5\ -3\ -2\ +1, \quad -5\ +3\ +2\ +1, \quad 4\ +3\ -3\ -2\ -1,$$
$$4\ -3\ -3\ +2\ +1, \quad -4\ +3\ +3\ -2\ +1,$$

which may be distinguished as a, b, c, d, e respectively. Then it is found that -8 is a product-sum peculiar to a; -9, -5 to b; 0 to c, and -3 to d (noting only those which belong to reduced solutions). These answer to the norms 2341; 2351, 2251; 1901; and 2141 respectively. It is easily verified that there are no q, r factors, and we conclude that they are all prime. On the other hand, -4, which gives the norm $(39+4)(39+15)-(15-4)^2 = 2201$, is common to a and e. This norm therefore contains two p factors; and, in fact,

$$2201 = 31 \times 71.$$

The analysis by which we justify this mode of discriminating a complex prime from other complex integers is a straightforward application of the methods and results of Arts. 8 and 29.*

Tables.

The tables which follow give the prime and reciprocal factors of every prime p, of the form $10\mu+1$, less than 10000, and of a few larger primes. They are arranged in seven columns. The first contains the value of p. The second contains the minimum values of X, Y which satisfy the equation

$$X^2 - 5Y^2 = 4p,$$

the sign of Y being fixed as explained in Art. 38. The third column contains the minimum residue of $3XY$, mod 5, and shows by what power of $\omega + \omega^{-1}$ the simplest prime factor must be multiplied to obtain the simplest primary factor. The next two columns contain the coordinates of the simplest prime factor and the simplest primary prime factor, arranged in the order

$$a_0,\ a_1,\ a_2,\ a_3,\ a_4.$$

It may be noted that in a former paper (*Proc. Lond. Math. Soc.*, Vol. XVIII., p. 229) the coordinates were tabulated with the indices in geometric, not arithmetic, progression a_1, a_2, a_4, a_3; a_0. The arith-

[* The question is the subject of a supplementary note which follows. But the discussion is not quite on the lines suggested above, for it was found convenient to consider the factors of $a\omega$ itself rather than the factors of the norm.]

metic order is here adopted as being more adapted for calculations. Whenever the simplest prime factor is itself primary ($Y \equiv 0$) the entry in the fifth column is merely $a\omega$.

The sixth column, except when p is less than 1000, contains the reciprocal factor of p; and the last column shows

$$Q = q_1 - q_2 - q_3 + q_4;$$

q_1, &c. being the coordinates entered in the sixth column, omitting the first entry, q_0.

For primes less than 1000 the reciprocal factors have been tabulated in the paper cited above, and it seemed useless to repeat them. The space thus saved has been used to give the prime factors published by Reuschle in his *Tafeln*, page 3. They are here expressed in terms of $a\omega$ the simplest prime factor of these tables, and $b\omega$ the simplest primary prime factor. The first entry is the "einfach," and the second is the "primär" of Reuschle. But when the primary factor is sufficiently simple, Reuschle does not give any entry under the *einfach*. In this column ϵ stands for $\omega + \omega^{-1}$, ϵ_2 for $\omega^2 + \omega^{-2}$, and Ω for $1 + \omega + \omega^2 + \omega^3 + \omega^4$. For example, $p = 11$, Reuschle's *einfach* factor

$$\begin{aligned} &= -\epsilon \,.\, a\omega^2 + \Omega \\ &= -(\omega + \omega^{-1})(\omega^2 + \omega^6 - \omega^8) + \Omega \\ &= -\omega^3 - \omega^2 + \omega^4 - \omega - 1 + \omega^2 + \Omega \\ &= \omega^2 + 2\omega^4; \end{aligned}$$

and the primary factor is $= -b\omega^2$

$$= \omega - \omega^3 - 2\omega^4.$$

It will be observed that, disregarding some misprints and neglecting the terms in Ω, Reuschle always gives the simplest primary prime factor. The *einfach* are the simplest factors for about half of the primes, but in some cases (241, 431) the primary factor is simpler than the *einfach*. I find four misprints.

$p = 601$.	*einfach.*	$-a^2$	is printed	instead of	$-a^3$.
$p = 751$.	*primär.*	$5a^2$	,,	,,	$8a^2$.
$p = 821$.	*primär.*	$4 + 4a -$ &c.	,,	,,	$1 + 4a -$ &c.
$p = 881$.	*primär.*	$-5a$	,,	,,	$-5a^2$.

[To ensure the accuracy of the tables, as far as practicable, the proof-sheets have been systematically tested and every entry has been checked at least once. Negative numbers are denoted by dots placed over them, a method which has the authority of Prof. Sylvester's example. Thus for $p = 211$, $Y = -6$, $a\omega = 3\omega - 2\omega^4$.]

p	X, Y	$3XY$	Prime Factors. Simplest, $=a\omega$.	Simplest primary $b\omega$	Reuschle's Factors.	Q
11	7, 1	1	0, 1, 0, 1, $\dot{1}$	0, 0, 2, $\dot{1}$, 1	$-\epsilon a\omega^2+\Omega$, $-b\omega^2$	5
31	12, 2	2	0, 2, $\dot{1}$, 0, 0	2, $\dot{3}$, 3, $\dot{1}$, 0	$a\omega^2$, $-b\omega^2+3\Omega$	5
41	13, 1	$\dot{1}$	1, $\dot{1}$, 0, 2, 0	$\dot{1}$, $\dot{1}$, 0, 1, 2	..., $b\omega^{-1}+\Omega$	5
61	17, $\dot{3}$	2	1, $\dot{2}$, 0, 1, 1	$\dot{2}$, 3, $\dot{1}$, 0, 1	$-a\omega+\Omega$, $-b\omega+\Omega$	5
71	17, $\dot{1}$	$\dot{1}$	2, 1, $\dot{1}$, 0, $\dot{1}$	$\dot{1}$, $\dot{1}$, 1, 3, 0	$-\epsilon a\omega^{-1}+\Omega$, $b\omega^{-1}+\Omega$	5
101	22, 4	$\dot{1}$	$\dot{2}$, 2, 0, 1, 0	1, 1, $\dot{2}$, 0, 2	..., $b\omega^{-1}-\Omega$	5
31	23, 1	$\dot{1}$	2, 0, 1, $\dot{2}$, 1	2, 2, $\dot{2}$, $\dot{1}$, 0	$a\omega^2-\Omega$, $-b\omega^2+2\Omega$	5
51	27, $\dot{5}$	0	1, 1, 2, $\dot{2}$, $\dot{1}$	$a\omega$	..., $-a\omega^2+\Omega$	$\dot{2}\dot{0}$
81	27, 1	1	$\dot{1}$, 0, $\dot{1}$, 0, 3	3, $\dot{2}$, 0, 2, $\dot{1}$	$\epsilon a\omega^3+\Omega$, $b\omega^3-2\Omega$	5
91	28, 2	$\dot{2}$	3, $\dot{1}$, $\dot{1}$, 1, 0	$\dot{3}$, 2, 4, 1, $\dot{2}$	$a\omega+\Omega$, $-b\omega-2\Omega$	5
* 211	32, $\dot{6}$	$\dot{1}$	0, 3, 0, 0, $\dot{2}$	0, $\dot{2}$, $\dot{2}$, 3, 3	$-a\omega^3$, $b\omega^3+2\Omega$	25
41	33, 5	0	2, 2, $\dot{1}$, 1, $\dot{2}$	$a\omega$	$\epsilon a\omega^2$, $-a\omega^2+2\Omega$	$\dot{2}\dot{0}$
51	32, $\dot{2}$	$\dot{2}$	$\dot{1}$, 2, 2, $\dot{2}$, 0	1, $\dot{4}$, $\dot{3}$, 3, 4	$-\epsilon_2 a\omega^2+2\Omega$, $b\omega^2+3\Omega$	$\dot{2}\dot{0}$
71	33, $\dot{1}$	1	$\dot{2}$, 0, 3, 0, 1	0, 0, 1, $\dot{3}$, 3	..., $b\omega^{-1}$	5
* 81	37, $\dot{7}$	$\dot{2}$	0, $\dot{3}$, 1, 1, 2	2, 6, 1, $\dot{4}$, $\dot{4}$	$-a\omega^{-1}+\Omega$, $b\omega^{-1}+4\Omega$	25
311	37, $\dot{5}$	0	3, $\dot{2}$, $\dot{1}$, 0, 1	$a\omega$	..., $a\omega^{-1}+2\Omega$	5
31	37, 3	$\dot{2}$	0, 3, $\dot{2}$, $\dot{1}$, 1	$\dot{3}$, $\dot{3}$, 3, 4, 0	$-\epsilon a\omega^3+\Omega$, $b\omega^3+3\Omega$	5
401	43, $\dot{7}$	2	2, 1, 0, $\dot{3}$, 2	1, 1, $\dot{2}$, 5, $\dot{3}$	$\epsilon_2 a\omega^3-2\Omega$, $-b\omega^3+\Omega$	5
* 21	42, $\dot{4}$	1	2, $\dot{3}$, 0, 0, 2	$\dot{1}$, 2, $\dot{3}$, 2, 2	..., $-b\omega^{-1}+2\Omega$	25
31	43, 5	0	$\dot{2}$, 3, 0, 2, $\dot{1}$	$a\omega$	$\epsilon_2 a\omega^{-1}-\Omega$, $-a\omega^{-1}+3\Omega$	$\dot{2}\dot{0}$
* 61	43, 1	$\dot{1}$	$\dot{1}$, 0, 4, $\dot{1}$, 0	$\dot{2}$, 2, 2, 2, $\dot{3}$	..., $-b\omega^2+2\Omega$	25
91	47, 7	2	2, $\dot{1}$, 1, $\dot{3}$, 2	$\dot{1}$, 4, $\dot{5}$, 6, $\dot{3}$	$-a\omega^3+2\Omega$, $-b\omega^3+6\Omega$	5
* 521	47, $\dot{5}$	0	0, 4, $\dot{1}$, $\dot{1}$, $\dot{1}$	$a\omega$	..., $a\omega^2+\Omega$	25
41	47, 3	$\dot{2}$	$\dot{3}$, 0, 0, 1, 3	4, 4, 0, $\dot{4}$, $\dot{3}$	$-a\omega^{-1}$, $-b\omega^{-1}+4\Omega$	5
71	48, 2	$\dot{2}$	0, 1, 1, 3, $\dot{3}$	4, $\dot{1}$, $\dot{4}$, $\dot{2}$, 5	$-a\omega^2+\Omega$, $b\omega^2+4\Omega$	$\dot{2}\dot{0}$
601	53, 9	1	0, 2, 0, $\dot{3}$, 3	$\dot{4}$, 1, 2, $\dot{2}$, 4	$-\epsilon_2 a\omega^{-1}+2\Omega$, $-b\omega^{-1}+\Omega$	5
31	52, 6	1	4, 0, $\dot{1}$, 0, $\dot{2}$	$\dot{2}$, 3, 0, $\dot{3}$, 4	$a\omega^2$, $-b\omega^2$	5
41	53, $\dot{7}$	2	$\dot{2}$, 2, 1, 3, $\dot{2}$	2, $\dot{3}$, 4, $\dot{4}$, 3	$-a\omega^3+3\Omega$, $b\omega^3+4\Omega$	$\dot{2}\dot{0}$
61	57, $\dot{1}\dot{1}$	$\dot{1}$	0, $\dot{1}$, $\dot{3}$, 3, 2	0, 5, 2, $\dot{1}$, $\dot{4}$	$-\epsilon_2 a\omega^2+2\Omega$, $-b\omega^2+2\Omega$	$\dot{4}\dot{5}$
* 91	53, 3	2	3, 0, $\dot{3}$, 2, 0	$\dot{3}$, 0, 5, $\dot{5}$, 5	$a\omega^{-1}$, $-b\omega^{-1}+5\Omega$	25
701	53, 1	$\dot{1}$	0, $\dot{2}$, 4, 1, $\dot{1}$	$\dot{4}$, 1, 2, 3, $\dot{1}$	$-a\omega^3+\Omega$, $-b\omega^3+3\Omega$	5
51	57, 7	2	$\dot{1}$, 1, $\dot{2}$, $\dot{1}$, 4	6, $\dot{4}$, 2, 3, $\dot{6}$	$-\epsilon_2 a\omega-\Omega$, $b\omega+6\Omega$	5
61	58, $\dot{8}$	$\dot{2}$	1, 2, $\dot{3}$, $\dot{1}$, 3	$\dot{5}$, 0, 7, 4, $\dot{4}$	$\epsilon a\omega^{-1}$, $-b\omega^{-1}$	5
811	57, $\dot{1}$	$\dot{1}$	0, $\dot{1}$, 2, 3, $\dot{3}$	5, 0, $\dot{3}$, $\dot{1}$, 1	..., $b\omega^{-1}$	5
21	58, $\dot{4}$	$\dot{1}$	3, 1, 2, $\dot{1}$, $\dot{3}$	0, 5, 1, $\dot{3}$, $\dot{2}$	$-\epsilon_2 a\omega^2$, $-b\omega^2+\Omega$	5
* 81	62, $\dot{8}$	2	0, 1, 2, 2, $\dot{4}$	$\dot{3}$, 1, 1, $\dot{4}$, 6	$-a\omega^3+2\Omega$, $-b\omega^3+\Omega$	25
911	67, $\dot{1}\dot{3}$	2	1, 3, 0, $\dot{4}$, 1	3, $\dot{2}$, $\dot{1}$, 5, $\dot{4}$	$-a\omega^3+\Omega$, $-b\omega^3+5\Omega$	$\dot{2}\dot{0}$
41	62, $\dot{4}$	1	$\dot{4}$, 2, 1, 2, 0	2, $\dot{3}$, 4, 1, $\dot{2}$	$-a\omega^2+2\Omega$, $-b\omega^2+4\Omega$	5
71	67, 11	1	$\dot{3}$, 3, 2, $\dot{2}$, 1	4, $\dot{1}$, 1, 3, $\dot{5}$	$\epsilon_2 a\omega^{-1}$, $b\omega^{-1}+\Omega$	$\dot{2}\dot{0}$
* 91	63, 1	$\dot{1}$	1, 2, 1, $\dot{4}$, 2	4, 3, $\dot{2}$, $\dot{2}$, $\dot{2}$	..., $b\omega^{-1}+2\Omega$	25

p	X, Y	$3XY$	Prime Factors. Simplest, $=a\omega$.	Prime Factors. Simplest primary.	Reciprocal Factor.	Q
1021	67, $\dot{9}$	1	2, $\dot{2}$, $\dot{3}$, 3, 1	$\dot{1}$, $\dot{1}$, 1, $\dot{2}$, 5	$\dot{4}$, $\dot{12}$, 24, 0, $\dot{9}$	$\dot{45}$
* 31	68, $\dot{10}$	0	3, 1, 1, $\dot{4}$, 1	$a\omega$	2, $\dot{12}$, $\dot{2}$, $\dot{12}$, 23	25
51	72, $\dot{14}$	1	$\dot{2}$, $\dot{1}$, $\dot{2}$, 4, 2	1, $\dot{4}$, 3, 0, 2	$\dot{6}$, $\dot{4}$, 2, 23, $\dot{16}$	$\dot{45}$
61	67, $\dot{7}$	$\dot{2}$	1, 4, $\dot{1}$, 0, $\dot{3}$	$\dot{2}$, $\dot{7}$, $\dot{1}$, 5, 6	0, 7, $\dot{12}$, $\dot{16}$, 20	55
91	67, $\dot{5}$	0	$\dot{1}$, 4, 0, 1, $\dot{3}$	$a\omega$	$\dot{2}$, 20, $\dot{9}$, $\dot{18}$, 8	55
*1151	68, 2	$\dot{2}$	$\dot{3}$, 1, 4, $\dot{1}$, 1	6, $\dot{1}$, $\dot{6}$, $\dot{1}$, 4	14, 15, $\dot{20}$, 0, $\dot{10}$	25
71	72, $\dot{10}$	0	0, 0, $\dot{4}$, 2, 3	$a\omega$	6, $\dot{22}$, $\dot{1}$, 20, $\dot{4}$	$\dot{45}$
81	73, $\dot{11}$	1	2, 4, $\dot{3}$, $\dot{1}$, 0	$\dot{3}$, 2, $\dot{2}$, 4, 0	$\dot{13}$, 14, 0, 16, $\dot{18}$	$\dot{20}$
1201	77, $\dot{15}$	0	1, 1, $\dot{3}$, $\dot{2}$, 4	$a\omega$	24, $\dot{4}$, $\dot{18}$, 3, $\dot{6}$	5
31	73, 9	1	2, $\dot{1}$, $\dot{3}$, 4, 0	2, 2, $\dot{2}$, 4, $\dot{5}$	2, 14, $\dot{25}$, $\dot{4}$, 12	55
91	72, $\dot{2}$	$\dot{2}$	2, 0, 0, $\dot{4}$, 3	$\dot{6}$, $\dot{1}$, 5, 6, $\dot{3}$	8, 20, $\dot{9}$, 2, $\dot{22}$	5
1301	73, 5	0	$\dot{4}$, 1, 3, 0, 2	$a\omega$	24, 6, $\dot{13}$, $\dot{2}$, $\dot{16}$	5
21	73, 3	2	0, $\dot{1}$, $\dot{2}$, 5, 0	$\dot{1}$, $\dot{1}$, 6, $\dot{7}$, 5	$\dot{4}$, 3, $\dot{16}$, $\dot{10}$, 26	55
61	82, 16	1	0, 1, 0, $\dot{4}$, 4	5, 0, $\dot{3}$, 4, $\dot{4}$	20, 12, $\dot{12}$, $\dot{1}$, $\dot{20}$	5
81	77, 9	$\dot{1}$	4, $\dot{2}$, 1, $\dot{3}$, 1	$\dot{2}$, $\dot{2}$, 5, 2, $\dot{1}$	$\dot{18}$, 24, 0, 6, $\dot{13}$	5
1451	77, 5	0	1, $\dot{4}$, 2, 3, $\dot{1}$	$a\omega$	24, $\dot{4}$, 7, $\dot{22}$, $\dot{6}$	5
71	77, $\dot{3}$	2	$\dot{2}$, 0, $\dot{3}$, 3, 3	5, $\dot{5}$, 6, $\dot{3}$, $\dot{2}$	$\dot{4}$, $\dot{12}$, $\dot{6}$, 30, $\dot{9}$	$\dot{45}$
81	77, 1	1	$\dot{1}$, 5, $\dot{1}$, 0, $\dot{2}$	3, $\dot{2}$, 5, $\dot{3}$, $\dot{1}$	2, 14, 0, $\dot{29}$, 12	55
*1511	83, 13	2	1, $\dot{2}$, 5, 0, $\dot{2}$	$\dot{5}$, 8, $\dot{7}$, 3, 3	$\dot{20}$, 26, 6, $\dot{9}$, $\dot{4}$	25
31	87, $\dot{17}$	$\dot{2}$	5, $\dot{2}$, $\dot{2}$, $\dot{1}$, 1	$\dot{8}$, 2, 8, 4, $\dot{5}$	$\dot{28}$, 14, 15, $\dot{4}$, 2	5
71	83, 11	$\dot{1}$	$\dot{2}$, 1, $\dot{3}$, 4, 2	0, $\dot{5}$, 1, 2, 3	6, 8, $\dot{1}$, $\dot{30}$, 16	55
*1601	82, 8	$\dot{2}$	4, 1, 0, 0, $\dot{4}$	$\dot{4}$, $\dot{5}$, 0, 5, 5	$\dot{16}$, 0, $\dot{20}$, 15, 20	25
21	83, 9	1	3, 0, 3, 0, $\dot{4}$	5, $\dot{5}$, 1, 2, $\dot{2}$	$\dot{4}$, 8, 24, $\dot{25}$, $\dot{4}$	5
1721	83, $\dot{1}$	1	$\dot{2}$, 0, $\dot{2}$, 5, 1	0, 5, $\dot{4}$, 2, $\dot{2}$	6, $\dot{2}$, $\dot{6}$, $\dot{25}$, 26	55
41	87, $\dot{11}$	$\dot{1}$	1, 5, $\dot{2}$, $\dot{1}$, $\dot{2}$	$\dot{3}$, $\dot{3}$, $\dot{1}$, 6, 3	8, 0, $\dot{4}$, $\dot{28}$, 23	55
1801	93, 17	$\dot{2}$	2, $\dot{2}$, $\dot{2}$, 5, $\dot{1}$	1, 6, 3, $\dot{5}$, $\dot{3}$	$\dot{6}$, 6, $\dot{28}$, 3, 24	55
11	88, 10	0	0, 5, $\dot{3}$, $\dot{1}$, 1	$a\omega$	$\dot{25}$, 12, $\dot{2}$, 24, $\dot{10}$	$\dot{20}$
31	87, 7	2	5, $\dot{3}$, $\dot{1}$, 0, 0	$\dot{8}$, 7, $\dot{2}$, $\dot{1}$, 5	$\dot{28}$, 14, 20, $\dot{9}$, 2	5
61	87, $\dot{5}$	0	3, 3, $\dot{1}$, 0, $\dot{4}$	$a\omega$	$\dot{10}$, 2, $\dot{22}$, $\dot{1}$, 30	55
* 71	92, 14	$\dot{1}$	2, 0, 2, 2, $\dot{5}$	4, $\dot{3}$, $\dot{3}$, 2, 2	16, 2, $\dot{33}$, 12, 2	25
1901	97, $\dot{19}$	1	3, 4, $\dot{2}$, $\dot{1}$, $\dot{3}$	1, 1, 3, $\dot{5}$, 2	$\dot{6}$, $\dot{4}$, $\dot{13}$, $\dot{12}$, 34	55
31	88, 2	$\dot{2}$	$\dot{1}$, 5, 0, $\dot{3}$, 1	$\dot{2}$, $\dot{7}$, 0, 7, 4	$\dot{18}$, $\dot{1}$, $\dot{20}$, 26, 12	5
* 51	93, $\dot{13}$	$\dot{2}$	2, $\dot{4}$, $\dot{1}$, 4, 1	1, 9, 4, $\dot{6}$, $\dot{6}$	$\dot{6}$, 0, $\dot{30}$, 20, 15	25
2011	93, $\dot{11}$	1	4, $\dot{4}$, $\dot{1}$, 1, $\dot{2}$	3, $\dot{2}$, 4, 0, $\dot{4}$	15, $\dot{18}$, $\dot{22}$, 24, 0	$\dot{20}$
81	98, 16	$\dot{1}$	$\dot{3}$, 5, 1, $\dot{2}$, 1	2, 2, 3, $\dot{1}$, $\dot{5}$	2, $\dot{6}$, $\dot{30}$, 26, 7	5
2111	92, $\dot{2}$	$\dot{2}$	$\dot{4}$, $\dot{1}$, 4, 0, 2	8, 3, $\dot{6}$, $\dot{5}$, 1	$\dot{20}$, $\dot{18}$, 8, $\dot{1}$, 30	5
31	93, $\dot{5}$	0	3, $\dot{2}$, 0, $\dot{3}$, 4	$a\omega$	2, $\dot{36}$, 15, 6, 12	$\dot{45}$
41	97, $\dot{13}$	2	$\dot{3}$, 4, 1, 2, $\dot{3}$	4, $\dot{6}$, 5, $\dot{4}$, 2	$\dot{12}$, 35, $\dot{4}$, $\dot{18}$, $\dot{2}$	55
61	93, $\dot{1}$	1	$\dot{1}$, $\dot{4}$, 4, 1, 2	3, $\dot{2}$, 4, $\dot{5}$, 1	$\dot{10}$, 12, 3, 14, $\dot{30}$	$\dot{45}$
*2221	98, 12	$\dot{2}$	5, $\dot{1}$, 2, $\dot{3}$, $\dot{1}$	$\dot{6}$, $\dot{3}$, 2, 7, 2	26, $\dot{13}$, 2, $\dot{28}$, 12	25
51	97, 9	$\dot{1}$	3, 2, 0, 1, $\dot{5}$	1, $\dot{4}$, $\dot{2}$, 5, 2	$\dot{6}$, 6, $\dot{33}$, 8, 24	55
81	102, $\dot{16}$	$\dot{1}$	4, $\dot{2}$, $\dot{4}$, 2, 1	$\dot{2}$, 3, 5, 2, $\dot{6}$	$\dot{18}$, $\dot{16}$, 35, $\dot{4}$, 2	$\dot{45}$
2311	107, 21	1	$\dot{5}$, 1, 0, 1, 4	5, $\dot{5}$, 2, 4, $\dot{4}$	$\dot{30}$, 2, $\dot{12}$, 24, 15	5
41	97, 3	$\dot{2}$	$\dot{3}$, 5, 0, 1, $\dot{2}$	4, $\dot{6}$, $\dot{5}$, 1, 7	$\dot{12}$, 30, 6, $\dot{28}$, 3	55
51	97, $\dot{1}$	$\dot{1}$	3, 2, $\dot{5}$, 1, 0	$\dot{4}$, 1, 3, 5, $\dot{3}$	$\dot{1}$, $\dot{34}$, 12, $\dot{2}$, 24	$\dot{20}$

[redacted]

p	X, Y	$3XY$	Prime Factors. Simplest, $=a\omega$.	Prime Factors. Simplest primary.	Reciprocal Factor.	Q
2371	103, 15	0	$\dot{1}, \dot{1}, 6, \dot{2}, 0$	$a\omega$	$\dot{4}, 38, \dot{1}\dot{6}, \dot{1}\dot{0}, \dot{9}$	55
81	98, $\dot{4}$	$\dot{1}$	$2, \dot{5}, 1, 3, 1$	$\dot{3}, \dot{3}, \dot{2}, 4, 5$	$\dot{1}\dot{3}, \dot{6}, \dot{2}\dot{0}, 36, 2$	$\dot{2}\dot{0}$
2411	107, 19	$\dot{1}$	$5, \dot{1}, 2, \dot{2}, \dot{3}$	$0, \dot{5}, 2, 4, 1$	$\dot{2}\dot{5}, 2, \dot{1}\dot{2}, 34, 0$	$\dot{2}\dot{0}$
41	103, $\dot{1}\dot{3}$	$\dot{2}$	$\dot{2}, \dot{1}, \dot{1}, 6, 0$	$7, 7, \dot{1}, \dot{9}, \dot{2}$	$\dot{2}, \dot{1}\dot{0}, \dot{9}, \dot{1}\dot{8}, 38$	55
2521	102, 8	$\dot{2}$	$\dot{2}, 1, \dot{4}, 2, 4$	$0, 5, 6, \dot{3}, \dot{7}$	$6, \dot{2}\dot{2}, \dot{1}\dot{6}, 35, \dot{4}$	$\dot{4}\dot{5}$
31	112, $\dot{2}\dot{2}$	$\dot{2}$	$0, \dot{2}, \dot{2}, \dot{1}, 6$	$\dot{3}, 7, 8, \dot{1}, \dot{1}\dot{0}$	$12, 14, \dot{4}\dot{0}, 6, 7$	55
51	103, 9	1	$0, 2, \dot{5}, 2, 3$	$\dot{4}, 6, \dot{3}, 3, \dot{1}$	$\dot{1}\dot{6}, 36, 2, \dot{2}\dot{2}, \dot{1}$	55
* 91	103, $\dot{7}$	2	$\dot{2}, 5, 2, \dot{3}, 0$	$7, \dot{5}, 0, 5, \dot{5}$	$\dot{2}, 1, \dot{3}\dot{6}, 24, 14$	25
2621	103, 5	0	$4, \dot{1}, \dot{4}, 3, 0$	$a\omega$	$16, 23, \dot{3}\dot{6}, 0, \dot{4}$	55
71	108, 14	1	$3, 0, 3, \dot{5}, 1$	$0, \dot{5}, 6, \dot{3}, 3$	$6, \dot{1}\dot{2}, \dot{6}, \dot{2}\dot{5}, 36$	55
2711	107, 11	1	$0, \dot{4}, 5, 1, \dot{1}$	$\dot{5}, 5, \dot{3}, 4, 1$	$30, \dot{2}\dot{8}, \dot{2}, \dot{1}\dot{6}, 15$	5
31	112, $\dot{1}\dot{8}$	2	$\dot{5}, 2, 4, 0, 0$	$7, \dot{3}, \dot{2}, 4, \dot{5}$	$\dot{3}\dot{3}, \dot{6}, 30, \dot{4}, 12$	$\dot{2}\dot{0}$
41	113, 19	1	$6, \dot{2}, 1, \dot{2}, \dot{1}$	$4, \dot{6}, 5, 1, \dot{3}$	$38, \dot{2}\dot{0}, \dot{4}, \dot{1}\dot{8}, 3$	5
91	108, $\dot{1}\dot{0}$	0	$\dot{3}, \dot{3}, 4, 1, 3$	$a\omega$	$8, 15, 6, 12, \dot{4}\dot{2}$	$\dot{4}\dot{5}$
2801	107, 7	2	$\dot{1}, 6, \dot{2}, \dot{1}, \dot{1}$	$6, \dot{9}, 7, \dot{2}, \dot{1}$	$14, 6, 7, \dot{4}\dot{2}, 14$	55
51	107, 3	$\dot{2}$	$4, \dot{3}, \dot{3}, 3, 0$	$\dot{4}, 6, 7, \dot{2}, \dot{6}$	$\dot{1}\dot{6}, \dot{9}, 42, \dot{1}\dot{2}, \dot{6}$	$\dot{4}\dot{5}$
61	107, 1	1	$0, 1, 5, \dot{4}, \dot{1}$	$0, 5, \dot{3}, 4, \dot{4}$	$0, \dot{2}\dot{8}, 23, 24, \dot{2}\dot{0}$	$\dot{9}\dot{5}$
2971	117, 19	$\dot{1}$	$\dot{3}, 6, \dot{1}, 0, \dot{1}$	$\dot{1}, \dot{1}, \dot{4}, 3, 5$	$\dot{4}, 23, 14, \dot{4}\dot{0}, 6$	55
*3001	122, 24	$\dot{1}$	$\dot{2}, 1, \dot{2}, \dot{2}, 6$	$\dot{4}, 4, 4, \dot{1}, \dot{1}$	$24, 0, \dot{4}\dot{0}, 15, 0$	25
11	112, $\dot{1}\dot{0}$	0	$\dot{2}, \dot{2}, \dot{1}, 0, 6$	$a\omega$	$0, 7, \dot{4}\dot{2}, 14, 20$	55
41	113, $\dot{1}\dot{1}$	1	$1, 3, \dot{4}, \dot{2}, 4$	$\dot{6}, 4, 0, 1, 2$	$\dot{4}\dot{2}, 20, 16, 2, 3$	5
61	117, $\dot{1}\dot{7}$	$\dot{2}$	$1, 4, \dot{1}, \dot{5}, 2$	$\dot{7}, \dot{7}, 4, 10, 1$	$35, 2, 8, \dot{1}\dot{6}, \dot{3}\dot{0}$	$\dot{2}\dot{0}$
3121	123, $\dot{2}\dot{3}$	$\dot{2}$	$0, 1, 6, \dot{2}, \dot{3}$	$4, \dot{6}, \dot{9}, 3, 10$	$\dot{9}, 38, \dot{6}, \dot{3}\dot{0}, 6$	80
81	113, $\dot{3}$	$\dot{2}$	$\dot{1}, 5, 0, 2, \dot{4}$	$3, \dot{7}, \dot{5}, 2, 9$	$2, \dot{3}\dot{6}, 20, 26, \dot{1}\dot{3}$	$\dot{9}\dot{5}$
91	113, 1	$\dot{1}$	$1, 4, 0, 2, \dot{5}$	$\dot{1}, 4, 5, \dot{4}, \dot{3}$	$\dot{2}, \dot{2}\dot{5}, 36, 12, \dot{2}\dot{2}$	$\dot{9}\dot{5}$
3221	122, 20	0	$5, 0, \dot{4}, 2, \dot{2}$	$a\omega$	$\dot{2}\dot{9}, \dot{2}\dot{2}, 24, 0, 26$	$\dot{2}\dot{0}$
* 51	127, 25	0	$1, 0, 5, \dot{5}, 0$	$a\omega$	$\dot{1}, \dot{2}\dot{0}, 20, 30, \dot{3}\dot{0}$	$\dot{1}\dot{0}\dot{0}$
71	117, $\dot{1}\dot{1}$	$\dot{1}$	$2, 1, \dot{1}, \dot{5}, 4$	$\dot{6}, \dot{1}, 6, 3, 0$	$26, 23, \dot{1}\dot{6}, 0, \dot{3}\dot{4}$	5
3301	118, $\dot{1}\dot{2}$	2	$2, 1, 5, \dot{3}, \dot{3}$	$\dot{4}, 6, \dot{7}, 5, 2$	$\dot{1}, 26, 2, \dot{4}\dot{2}, 14$	80
31	123, $\dot{1}\dot{9}$	$\dot{1}$	$2, 5, \dot{4}, \dot{2}, 1$	$7, 2, \dot{2}, \dot{6}, 0$	$\dot{3}\dot{3}, 24, 0, 26, \dot{1}\dot{8}$	$\dot{2}\dot{0}$
61	117, 7	2	$1, \dot{2}, 5, 1, \dot{4}$	$\dot{7}, 8, \dot{6}, 0, 6$	$35, \dot{2}\dot{8}, \dot{2}\dot{2}, 14, 0$	$\dot{2}\dot{0}$
91	117, 5	0	$\dot{1}, \dot{1}, 0, 6, \dot{3}$	$a\omega$	$23, 20, \dot{4}, 2, \dot{4}\dot{2}$	$\dot{2}\dot{0}$
3461	118, 4	1	$\dot{1}, 6, \dot{1}, 1, \dot{3}$	$\dot{2}, 3, \dot{6}, 5, 1$	$0, \dot{2}\dot{8}, 8, 39, \dot{2}\dot{0}$	$\dot{9}\dot{5}$
91	128, 22	$\dot{2}$	$3, \dot{1}, 4, 1, \dot{5}$	$2, \dot{3}, \dot{6}, 1, 8$	$\dot{2}\dot{2}, 15, 36, \dot{2}\dot{8}, \dot{2}$	5
3511	132, $\dot{2}\dot{6}$	$\dot{1}$	$0, \dot{6}, 2, 3, 2$	$5, 5, 2, \dot{6}, \dot{4}$	$20, 2, 8, \dot{4}\dot{6}, 15$	55
41	122, $\dot{1}\dot{2}$	$\dot{2}$	$\dot{3}, 0, 0, 6, \dot{2}$	$9, 4, \dot{5}, \dot{9}, 2$	$3, 30, \dot{4}, 12, \dot{4}\dot{2}$	$\dot{2}\dot{0}$
* 71	123, 13	2	$\dot{5}, 2, 4, \dot{1}, 2$	$9, \dot{3}, \dot{3}, 7, \dot{8}$	$36, 12, \dot{3}\dot{3}, 2, \dot{1}\dot{8}$	25
81	127, $\dot{1}\dot{9}$	1	$4, \dot{5}, \dot{1}, 0, 3$	$\dot{2}, 3, \dot{5}, 2, 4$	$\dot{1}\dot{8}, 34, 15, \dot{3}\dot{4}, 2$	55
3631	123, 11	$\dot{1}$	$\dot{3}, 0, 6, \dot{2}, 1$	$\dot{3}, 2, 3, 4, \dot{5}$	$12, 39, \dot{3}\dot{0}, \dot{4}, \dot{1}\dot{8}$	55
71	127, $\dot{1}\dot{7}$	$\dot{2}$	$\dot{2}, 6, 1, \dot{3}, \dot{1}$	$0, \dot{1}\dot{0}, \dot{4}, 7, 8$	$6, 8, 34, \dot{4}\dot{0}, \dot{9}$	5
91	128, $\dot{1}\dot{8}$	$\dot{2}$	$3, \dot{1}, \dot{1}, \dot{4}, 5$	$\dot{8}, 2, 9, 6, \dot{7}$	$\dot{1}\dot{2}, \dot{4}\dot{0}, 26, 2, 23$	$\dot{4}\dot{5}$
3701	122, 4	$\dot{1}$	$\dot{2}, 2, 5, \dot{4}, 0$	$1, \dot{4}, \dot{2}, 0, 7$	$\dot{6}, \dot{2}\dot{9}, 42, 8, \dot{1}\dot{6}$	$\dot{9}\dot{5}$
61	133, $\dot{2}\dot{3}$	$\dot{2}$	$6, 2, \dot{3}, \dot{1}, \dot{2}$	$\dot{1}\dot{0}, \dot{5}, 7, 9, 1$	$35, 12, \dot{2}, \dot{6}, \dot{4}\dot{0}$	$\dot{2}\dot{0}$
3821	127, 13	$\dot{2}$	$3, \dot{4}, 1, \dot{3}, 4$	$\dot{5}, 5, 6, 2, \dot{7}$	$\dot{9}, 48, \dot{1}\dot{6}, \dot{2}\dot{0}, \dot{4}$	80
* 51	128, $\dot{1}\dot{4}$	$\dot{1}$	$\dot{5}, 1, 0, 5, 1$	$\dot{4}, \dot{5}, 5, 5, 0$	$34, \dot{2}\dot{5}, 0, \dot{3}\dot{0}, 20$	25

			Prime Factors.			
p	X, Y	$3XY$	Simplest, $=a\omega$.	Simplest primary.	Reciprocal Factor.	Q
3881	127, 11	1	1̇, 5̇, 4, 0, 3	2̇, 3, 5̇, 7, 1̇	7, 6̇, 1̇0̇, 3̇4̇, 42	80
3911	137, 25	0	2̇, 3, 1̇, 5, 4̇	$a\omega$	0, 47, 2̇2̇, 6, 2̇0̇	55
31	127, 9	1̇	6̇, 3, 1, 2, 1	3, 3, 5̇, 3̇, 4	4̇8̇, 24, 15, 6, 2	5
4001	127, 5	0	1, 1, 2, 3, 6̇	$a\omega$	24, 16, 4̇8̇, 8, 1̇	55
* 21	127, 3̇	2	3, 1̇, 0, 5̇, 4	0, 4, 6̇, 9, 6̇	6, 42, 1̇8̇, 2, 3̇3̇	25
51	128, 6̇	1	5, 3̇, 0, 3̇, 3	1, 4̇, 7, 2̇, 1̇	24, 4̇9̇, 2, 8, 14	4̇5̇
91	128, 2	2̇	3, 4, 1̇, 1, 5̇	3̇, 8̇, 1̇, 6, 8	8, 1̇0̇, 36, 7, 4̇2̇	9̇5̇
4111	132, 1̇4̇	1	0, 4̇, 0, 6, 1̇	5̇, 0, 2̇, 1̇, 6	30, 2, 23, 1̇6̇, 4̇0̇	4̇5̇
4201	143, 2̇7̇	2	3̇, 4̇, 5, 2, 2	1, 6, 7̇, 5, 3̇	6̇, 26, 2, 23, 4̇6̇	4̇5̇
11	133, 13	2	4̇, 0, 1̇, 1, 6	10, 5̇, 2, 4, 9̇	15, 2, 38, 3̇6̇, 2̇0̇	2̇0̇
31	132, 1̇0̇	0	2, 2, 3, 6̇, 0	$a\omega$	2, 1̇, 20, 26, 4̇8̇	9̇5̇
41	137, 1̇9̇	1	1, 2, 4̇, 3̇, 5	7, 3̇, 1̇, 1, 2̇	48, 1̇0̇, 2̇9̇, 2, 1̇2̇	5
61	138, 20	0	5̇, 5, 2, 1̇, 1	$a\omega$	30, 1̇3̇, 4̇2̇, 24, 0	5
71	133, 1̇1̇	1	3, 5, 2̇, 0, 4̇	0, 0, 4̇, 7, 2̇	6, 2̇, 24, 20, 4̇9̇	9̇5̇
*4391	133, 5̇	0	2, 0, 5, 0, 5̇	$a\omega$	2̇2̇, 24, 34, 3̇6̇, 1̇	25
4421	133, 1	1̇	3, 1, 2, 6̇, 2	5, 5, 4̇, 3̇, 2̇	4̇, 2̇2̇, 6̇, 2̇0̇, 51	55
* 41	138, 16	1̇	1, 3̇, 1, 6, 3̇	6̇, 2̇, 3, 3, 3	4̇2̇, 6̇, 6̇, 14, 39	25
51	143, 2̇3̇	2̇	2, 3, 2̇, 5̇, 4	9̇, 4̇, 8, 10, 3̇	4̇6̇, 9̇, 12, 8, 34	5
* 81	137, 13	2̇	5, 2, 4̇, 1, 3̇	8̇, 4̇, 6, 6, 1	2̇8̇, 2̇2̇, 3, 2̇, 48	25
4561	143, 21	1̇	6̇, 2, 3, 0, 3	2̇, 2̇, 4, 5, 4̇	50, 2, 2̇2̇, 6̇, 2̇5̇	5
91	137, 9	1̇	1, 5̇, 3, 4, 2̇	7, 2, 1̇, 4̇, 2̇	48, 1̇0̇, 6, 3̇3̇, 1̇2̇	5
*4621	147, 25	0	5̇, 4, 1̇, 4, 1̇	$a\omega$	3̇4̇, 47, 2, 2, 1̇8̇	25
* 51	152, 30	0	6, 5̇, 0, 0, 0	$a\omega$	3̇6̇, 30, 30, 2̇5̇, 0	25
* 91	137, 1	1	6, 4̇, 1̇, 1̇, 1	3̇, 5, 5̇, 0, 5	4̇2̇, 34, 24, 1̇6̇, 1̇	25
4721	142, 1̇6̇	1̇	2, 1, 4, 0, 6̇	4, 6̇, 4̇, 3, 5	16, 1̇2̇, 4̇6̇, 35, 6	5
* 51	143, 1̇7̇	2	2, 6̇, 1, 1, 4	4̇, 9, 6̇, 4, 1̇	1̇, 3̇0̇, 0, 50, 2̇0̇	1̇0̇0̇
4801	153, 2̇9̇	1̇	5, 2̇, 1̇, 4, 4	6, 1, 8̇, 2̇, 4	14, 5̇4̇, 12, 3, 24	4̇5̇
31	143, 15	0	3, 3, 0, 2, 6̇	$a\omega$	2, 1̇6̇, 50, 4̇, 3̇3̇	9̇5̇
61	142, 12	2	1, 2̇, 0, 4̇, 6	3, 3, 6̇, 10, 9̇	15, 42, 4̇2̇, 6̇, 1̇0̇	80
* 71	148, 2̇2̇	2	0, 7, 1̇, 1̇, 3̇	4, 8̇, 7, 3̇, 2	9̇, 1̇8̇, 2, 52, 2̇8̇	1̇0̇0̇
4931	152, 2̇6̇	1̇	1̇, 2̇, 6, 2, 4̇	8, 2̇, 5̇, 3̇, 4	47, 3̇6̇, 2̇0̇, 6, 2	2̇0̇
51	147, 19	1̇	2̇, 2, 5̇, 1, 5	4̇, 6, 3, 0, 3̇	24, 3̇4̇, 2̇8̇, 38, 1̇	4̇5̇
5011	143, 9	1	1̇, 6, 1̇, 4̇, 2	7̇, 3, 1̇, 0, 6	4̇0̇, 2, 2̇2̇, 39, 20	5
21	142, 4	1̇	3̇, 4̇, 4, 0, 4	4, 4, 1, 7̇, 0	9̇, 1̇2̇, 1̇6̇, 2̇0̇, 56	80
51	143, 7	2̇	7, 2̇, 2̇, 0, 1̇	9̇, 1, 8, 5, 3̇	54, 9̇, 2̇8̇, 2̇, 1̇6̇	5
* 81	143, 5	0	2̇, 1, 4̇, 1, 6	$a\omega$	1̇8̇, 23, 38, 4̇2̇, 2̇	25
5101	143, 3	2	3̇, 6, 0, 2, 3̇	6, 9̇, 8, 5̇, 2	14, 4̇9̇, 2, 38, 6̇	9̇5̇
71	152, 2̇2̇	2̇	2̇, 4̇, 6, 2, 1̇	10, 5, 9̇, 8̇, 3	36, 4̇2̇, 1̇, 2̇0̇, 26	5
5231	148, 1̇4̇	1̇	3̇, 5, 1, 3, 4̇	3̇, 2, 8, 1̇, 5̇	12, 5̇6̇, 15, 26, 2	9̇5̇
61	158, 28	2	1, 5, 6̇, 1, 1	5, 1̇0̇, 12, 6̇, 1	3̇0̇, 47, 2̇, 14, 3̇0̇	5
* 81	162, 32	2	0, 4̇, 2, 3̇, 6	2, 6, 9̇, 11, 9̇	2, 38, 4̇2̇, 2̇2̇, 23	125
5351	148, 10	0	1, 4̇, 3, 5, 3̇	$a\omega$	5̇6̇, 6, 2, 23, 24	5
* 81	157, 25	0	3̇, 1, 6, 4̇, 1	$a\omega$	1̇3̇, 4̇2̇, 48, 8, 2̇	1̇0̇0̇
5431	148, 6̇	1	3̇, 1̇, 7, 1̇, 0	2, 3̇, 3, 6̇, 5	2, 54, 2̇5̇, 4̇, 2̇8̇	55
41	163, 3̇1̇	1	4̇, 3, 6, 2̇, 1̇	1̇, 1̇, 0, 4̇, 7	23, 30, 5̇4̇, 2, 2̇	80

p	X, Y	$3XY$	Prime Factors.		Reciprocal Factor.	Q
			Simplest, $=a\omega$.	Simplest primary.		
5471	148, 2̇	2	0, 1̇, 3, 5, 5̇	6̇, 4, 1, 7̇, 10	4̇9̇, 1̇2̇, 34, 0, 26	2̇0̇
5501	157, 2̇3̇	2	1̇, 1, 3, 4, 6̇	4̇, 1, 2, 7̇, 9	34, 26, 4̇8̇, 3, 1̇6̇	55
21	158, 2̇4̇	1̇	3, 6, 3̇, 1̇, 3̇	5, 5, 1, 8̇, 2̇	4̇, 8, 14, 35, 5̇4̇	9̇5̇
* 31	152, 1̇4̇	1	6̇, 4, 2, 2, 1̇	3, 4̇, 6, 1, 4̇	4̇8̇, 38, 23, 1̇2̇, 2̇	25
81	162, 2̇8̇	2	5, 2, 6̇, 0, 0	3̇, 3̇, 8, 6̇, 5	1̇3̇, 4̇6̇, 20, 4̇, 42	2̇0̇
* 91	163, 2̇9̇	1̇	1, 2, 6, 4̇, 3̇	1̇, 8, 3, 2̇, 7̇	2, 39, 1̇6̇, 4̇6̇, 24	125
*5641	153, 1̇3̇	2̇	2̇, 3̇, 0, 0, 7	2, 10, 5, 5̇, 1̇0̇	3, 1̇6̇, 54, 6̇, 3̇6̇	1̇0̇0̇
51	152, 1̇0̇	0	4̇, 6, 2, 2̇, 1̇	$a\omega$	1̇6̇, 16, 47, 4̇2̇, 6̇	5
5701	153, 11	1̇	0, 3, 6̇, 1, 4	6, 4̇, 3̇, 2̇, 4	3̇6̇, 51, 12, 2̇2̇, 6̇	55
11	157, 1̇9̇	1	5̇, 1, 0, 6, 1̇	0, 5̇, 7, 1̇, 1	2̇5̇, 42, 2̇, 24, 4̇0̇	2̇0̇
41	158, 20	0	3̇, 7, 1̇, 1, 2̇	$a\omega$	8, 4̇0̇, 9̇, 52, 1̇2̇	9̇5̇
91	153, 7	2̇	3, 4, 6̇, 1, 0	8̇, 3̇, 9, 6, 2̇	1̇2̇, 50, 9̇, 12, 4̇2̇	5
*5801	157, 17	2	6̇, 5, 1, 1, 0	11, 1̇0̇, 5, 0, 5̇	4̇6̇, 35, 30, 2̇0̇, 0	25
21	153, 5	0	6̇, 4, 1, 3, 0	$a\omega$	51, 4̇2̇, 1̇6̇, 0, 6	2̇0̇
51	153, 1̇	1	0, 3̇, 0, 7, 2̇	6, 1, 3̇, 3, 6̇	3̇6̇, 9̇, 1̇8̇, 8, 54	55
61	163, 2̇5̇	0	0, 0, 7, 1̇, 4̇	$a\omega$	2̇5̇, 52, 8, 3̇6̇, 0	80
* 81	157, 1̇5̇	0	3̇, 1, 1, 6, 4̇	$a\omega$	12, 48, 2̇2̇, 3, 4̇2̇	25
5981	168, 2̇3̇	2̇	1̇, 0, 5̇, 2, 6	2̇, 8, 10, 3̇, 1̇1̇	1̇8̇, 34, 30, 4̇9̇, 2	55
*6011	157, 1̇1̇	1̇	5, 3, 5̇, 0, 2̇	5̇, 2̇, 3, 8, 2̇	2̇0̇, 3̇4̇, 6, 9̇, 56	25
91	163, 21	1̇	1, 6̇, 5, 2, 0	6̇, 1̇, 0, 6, 2	4̇2̇, 20, 4̇, 47, 2̇2̇	4̇5̇
*6101	157, 7	2	1̇, 0, 1, 6, 5̇	4̇, 0, 5, 1̇0̇, 10	34, 35, 3̇0̇, 0, 4̇0̇	25
21	173, 3̇3̇	2̇	5, 4̇, 4̇, 3, 2	6̇, 9, 11, 2̇, 1̇0̇	26, 8, 6̇1̇, 20, 6	55
31	157, 5̇	0	7, 3̇, 2̇, 1̇, 0	$a\omega$	5̇8̇, 24, 30, 4̇, 7	5
51	157, 3̇	2	4, 1, 3, 1̇, 6̇	9̇, 6, 3̇, 2̇, 9	6̇, 9̇, 4̇8̇, 48, 14	5
*6211	168, 2̇6̇	1	1̇, 4, 5, 5̇, 1̇	2̇, 3̇, 2, 3̇, 7	0, 26, 5̇4̇, 9̇, 36	125
21	158, 4	1	2̇, 5, 3, 5̇, 1	5̇, 0, 1, 3̇, 8	9̇, 8, 5̇6̇, 20, 36	80
* 71	172, 30	0	0, 4, 1̇, 4, 6̇	$a\omega$	6, 47, 4̇8̇, 1̇8̇, 12	125
6301	177, 35	0	1, 1, 2, 7̇, 4	$a\omega$	24, 26, 2, 8, 6̇1̇	4̇5̇
11	167, 23	2̇	4̇, 4, 1̇, 5, 3̇	8, 2̇, 6̇, 5̇, 6	2̇0̇, 62, 1̇2̇, 6̇, 2̇5̇	55
61	163, 1̇5̇	0	0, 5, 2, 6̇, 1	$a\omega$	2̇5̇, 2, 4̇2̇, 14, 50	80
6421	167, 21	1	3̇, 2̇, 7, 2̇, 1	1̇, 4, 4̇, 8, 5̇	4̇, 5̇7̇, 24, 0, 36	4̇5̇
51	168, 2̇2̇	2	7, 1, 0, 3̇, 3̇	9̇, 6, 2̇, 0, 7	54, 9̇, 12, 4̇2̇, 1̇6̇	5
81	162, 8̇	2	0, 2, 6̇, 0, 5	7, 8̇, 8, 1̇, 5̇	42, 4̇6̇, 2̇5̇, 26, 2	4̇5̇
* 91	163, 1̇1̇	1	6, 1, 2, 3̇, 4̇	4, 7̇, 3, 3, 2̇	38, 1̇, 24, 5̇6̇, 6̇	25
6521	167, 1̇9̇	1	2, 3, 2, 7̇, 1	4, 4, 4̇, 3, 5̇	16, 8, 24, 15, 6̇4̇	9̇5̇
51	172, 26	1	2̇, 4, 3, 6̇, 2	6, 1, 2̇, 5, 8̇	14, 9̇, 52, 1̇2̇, 4̇6̇	9̇5̇
71	173, 2̇7̇	2	0, 6̇, 3, 5, 0	6̇, 9, 4̇, 2̇, 5	4̇9̇, 8, 1̇6̇, 50, 6	2̇0̇
81	163, 7	2̇	4, 0, 5, 3̇, 4̇	2̇, 7̇, 5̇, 7, 9	7, 24, 20, 6̇4̇, 12	80
6661	182, 36	1	0, 1, 0, 6, 6̇	5̇, 0, 7, 6̇, 6	30, 42, 4̇2̇, 1̇, 3̇0̇	55
91	167, 15	0	4, 4, 5̇, 1, 3̇	$a\omega$	1̇2̇, 3̇0̇, 4̇, 1̇8̇, 63	55
6701	173, 2̇5̇	0	1, 6, 2̇, 5̇, 2	$a\omega$	5̇6̇, 6, 1̇3̇, 38, 24	5
61	167, 13	2̇	4̇, 1̇, 1̇, 0, 7	3, 8, 4, 5̇, 9̇	1̇0̇, 2, 5̇7̇, 34, 30	55
81	177, 2̇9̇	1	1̇, 5̇, 4, 5, 2̇	7̇, 3, 0, 2, 4	62, 1̇6̇, 0, 3̇4̇, 1̇3̇	5
91	172, 2̇2̇	2̇	7, 0, 0, 4̇, 2̇	1̇1̇, 6̇, 5, 11, 2	5̇7̇, 20, 4̇, 42, 2̇	2̇0̇
*6841	183, 3̇5̇	0	7, 0, 5̇, 0, 0	$a\omega$	48, 24, 3̇6̇, 1̇, 3̇6̇	25

p	X, Y	$3XY$	Prime Factors. Simplest, $=a\omega$.	Prime Factors. Simplest primary.	Reciprocal Factor.	Q
*6871	167, 9	1̇	3̇, 0, 3̇, 7, 0	4, 7, 3̇, 3̇, 3̇	9̇, 12, 12, 42, 5̇8̇	1̇0̇0̇
6911	167, 7̇	2̇	1, 1̇, 6̇, 5, 2	2̇, 8, 9, 5̇, 9̇	0, 4̇8̇, 38, 34, 2̇5̇	1̇4̇5̇
61	167, 3̇	2	1, 2̇, 5̇, 6, 1	2̇, 2̇, 9, 1̇0̇, 6	0, 3̇3̇, 48, 24, 4̇0̇	1̇4̇5̇
71	167, 1	1	2, 7̇, 2, 3, 1	6̇, 4, 4̇, 3, 5	26, 8, 24, 6̇5̇, 6	55
91	172, 18	2̇	2, 5̇, 0, 6, 2̇	4, 9, 0, 9̇, 3̇	38, 0, 36, 3̇3̇, 4̇2̇	4̇5̇
7001	182, 3̇2̇	2̇	4, 2, 2, 7̇, 0	9̇, 9̇, 2, 13, 4	6̇, 16, 12, 38, 6̇1̇	9̇5̇
7121	173, 17	2̇	5̇, 1, 6, 2̇, 2	9, 1̇, 9̇, 2̇, 5	36, 38, 4̇6̇, 0, 2̇9̇	55
51	172, 14	1̇	2̇, 2, 0, 6, 5̇	6, 1, 7̇, 0, 2	14, 56, 3̇3̇, 2, 3̇6̇	55
*7211	187, 35	0	2̇, 2, 3̇, 3̇, 7	*aω*	50, 4, 5̇4̇, 16, 9̇	25
7321	183, 2̇9̇	1̇	7̇, 1, 2, 4, 2	5̇, 5̇, 6, 7, 2̇	66, 2̇2̇, 6̇, 3̇0̇, 9̇	5
31	173, 11	1̇	7, 0, 1, 2̇, 4̇	2, 7, 2̇, 6̇, 0	52, 6̇, 20, 4̇9̇ 1̇8̇	5
51	172, 6̇	1̇	2̇, 7, 0, 4̇, 0	4̇, 4̇, 2̇, 5, 7	24, 6, 42, 57, 1̇6̇	5
7411	192, 38	2̇	4̇, 4, 4, 5̇, 2	3, 7̇, 6̇, 5, 6	1̇0̇, 1̇8̇, 68, 1̇6̇, 2̇5̇	9̇5̇
* 51	173, 5̇	0	1, 4, 1̇, 6̇, 4	*aω*	5̇6̇, 1̇0̇, 0, 15, 50	25
81	173, 1	1̇	2, 5̇, 1, 2̇, 6	2, 3̇, 7̇, 4, 5	2, 5̇6̇, 35, 36, 1̇8̇	1̇4̇5̇
7541	187, 3̇1̇	1̇	6, 5̇, 2̇, 1̇, 3	3̇, 2, 9, 1, 7̇	4̇2̇, 50, 26, 3̇3̇, 2̇	55
61	182, 2̇4̇	1	0, 6, 0, 1, 6̇	0, 0, 7, 6̇, 1	0, 42, 4̇2̇, 3̇6̇, 35	155
91	183, 2̇5̇	0	2, 2, 4, 1, 7̇	*aω*	2̇2̇, 0, 66, 3̇3̇, 1̇2̇	4̇5̇
7621	177, 13	2̇	7̇, 1, 1, 2, 4	10, 5, 4̇, 8̇, 2̇	6̇4̇, 8, 6̇, 35, 26	5
81	177, 1̇1̇	1̇	1̇, 2̇, 4̇, 7, 1	3, 8, 0, 3̇, 6̇	2, 1̇6̇, 35, 36, 5̇8̇	1̇4̇5̇
91	187, 2̇9̇	1	6, 2, 1, 3̇, 5̇	3̇, 7, 1̇, 4̇, 3	4̇2̇, 0, 3̇4̇, 52, 23	5
7741	178, 1̇2̇	2	3, 2, 1, 7̇, 3	2, 2, 6̇, 11, 7̇	2̇2̇, 3̇0̇, 4, 1̇3̇, 68	55
7841	178, 8	2	7̇, 2, 1, 3, 3	12, 8̇, 4, 1, 7̇	68, 2̇0̇, 4, 3̇3̇, 1̇2̇	5
7901	178, 4̇	1̇	5, 3, 6̇, 1, 1̇	6, 1, 3̇, 7̇, 4	14, 51, 1̇8̇, 8, 5̇6̇	5
51	193, 33	2	2, 4̇, 0, 7, 3̇	9̇, 6, 3, 1̇0̇, 12	4̇6̇, 4, 2̇8̇, 23, 54	55
*8011	183, 17	2̇	6, 0, 3, 2̇, 5̇	5̇, 7̇, 2̇, 8, 8	30, 6, 36, 6̇4̇, 9̇	25
81	187, 2̇3̇	2	0, 7, 1̇, 0, 5̇	2, 8̇, 8, 6̇, 5	2, 34, 2̇5̇, 5̇4̇, 42	155
8101	182, 12	2	6̇, 1, 2̇, 4, 4	11, 9̇, 7, 2̇, 6̇	4̇6̇, 6, 1̇8̇, 63, 6̇	4̇5̇
11	193, 3̇1̇	1	1̇, 4̇, 4, 6, 3̇	8, 2̇, 1̇, 0, 4̇	7̇0̇, 7, 8, 24, 30	5
61	183, 13	2	4̇, 5, 4, 4̇, 1	10, 5̇, 3̇, 9, 9̇	15, 12, 7̇2̇, 24, 20	80
* 71	197, 3̇5̇	0	5, 1̇, 1̇, 6̇, 4	*aω*	4̇, 62, 1̇8̇, 7, 4̇8̇	25
91	187, 2̇1̇	1̇	1, 5, 2̇, 6̇, 3	8̇, 3̇, 4, 6, 3	63, 0, 6, 2̇8̇, 4̇2̇	2̇0̇
8221	183, 11	1̇	2̇, 4̇, 2, 1̇, 7	0, 5̇, 4̇, 7, 3	6, 2̇2̇, 54, 15, 5̇4̇	1̇4̇5̇
31	188, 22	2̇	1̇, 0, 5̇, 7, 1	3, 8, 5, 8̇, 6̇	2, 24, 3̇0̇, 4̇9̇, 52	155
91	187, 1̇9̇	1	1, 7, 4̇, 3̇, 0	7, 3̇, 4, 4̇, 2̇	48, 2̇0̇, 6, 5̇8̇, 23	55
8311	183, 7	2̇	1, 2, 2, 4, 7̇	5, 5̇, 8̇, 1̇, 11	3̇0̇, 3̇3̇, 68, 6̇, 0	9̇5̇
8431	188, 18	2	1̇, 8, 3̇, 1̇, 1̇	8, 1̇2̇, 10, 3̇, 1̇	2̇8̇, 6, 1̇0̇, 71, 2̇8̇	9̇5̇
61	187, 1̇5̇	0	3, 2̇, 1̇, 5̇, 6	*aω*	15, 62, 4̇2̇, 6̇, 3̇0̇	80
8501	197, 31	1	7̇, 4, 3, 1̇, 2	6, 4̇, 3, 5, 8̇	6̇1̇, 6, 52, 1̇2̇, 14	2̇0̇
21	198, 3̇2̇	2	5̇, 1̇, 2̇, 5, 5	9, 6̇, 6, 2̇, 5̇	36, 3, 24, 7̇0̇, 6	55
* 81	202, 36	1	4, 6̇, 2, 3̇, 4	2̇, 6, 9̇, 6, 1	1̇8̇, 68, 1̇2̇, 4̇2̇, 3	125
8641	187, 9	1̇	1, 0, 3, 4, 7̇	7, 3̇, 6̇, 1, 3	48, 15, 6̇4̇, 12, 1̇2̇	55
81	187, 7	2	5̇, 3̇, 4, 0, 5	7, 2, 7̇, 9, 1̇0̇	3̇3̇, 1̇6̇, 2̇0̇, 4, 72	80
8731	187, 3̇	2	0, 3̇, 4, 5, 5̇	8̇, 7, 2̇, 6̇, 10	72, 6̇, 3̇0̇, 9, 2̇8̇	5
41	187, 1	1	1, 3̇, 4̇, 7, 0	3̇, 3̇, 4, 4̇, 8	8, 2̇0̇, 56, 12, 5̇7̇	1̇4̇5̇

p	X, Y	$3XY$	Prime Factors. Simplest, $=a\omega$.	Prime Factors. Simplest primary.	Reciprocal Factor.
8761	193, 21	$\dot{1}$	$\dot{6}$, 2, $\dot{2}$, 5, 3	$\dot{2}$, $\dot{7}$, 4, 5, 1	50, $\dot{18}$, 8, $\dot{61}$, 20
8821	198, $\dot{28}$	$\dot{2}$	5, 1, 1, $\dot{7}$, 2	$\dot{11}$, $\dot{6}$, 6, 13, 0	6, $\dot{42}$, $\dot{1}$, $\dot{30}$, 66
* 31	188, 2	$\dot{2}$	4, 3, 1, 1, $\dot{7}$	$\dot{2}$, $\dot{9}$, $\dot{4}$, 6, 11	7, $\dot{12}$, 68, $\dot{22}$, 4
* 61	193, $\dot{19}$	$\dot{1}$	$\dot{1}$, 5, 4, $\dot{6}$, 0	3, 7, 2, $\dot{3}$, $\dot{8}$	$\dot{10}$, 16, $\dot{64}$, 6, 51
8941	203, $\dot{33}$	$\dot{2}$	8, $\dot{1}$, $\dot{1}$, $\dot{4}$, 0	$\dot{13}$, $\dot{3}$, 9, 11, $\dot{2}$	68, $\dot{40}$, $\dot{9}$, $\dot{28}$, 8
51	193, 17	$\dot{2}$	2, $\dot{7}$, 3, 0, 4	1, 11, 3, $\dot{5}$, $\dot{8}$	$\dot{6}$, $\dot{34}$, 7, 68, $\dot{36}$
71	192, 14	$\dot{1}$	2, 6, $\dot{6}$, 0, $\dot{1}$	$\dot{6}$, $\dot{1}$, 1, 8, 0	26, $\dot{42}$, $\dot{1}$, $\dot{40}$, 50
9001	207, 37	2	4, 1, $\dot{7}$, 4, $\dot{1}$	$\dot{4}$, $\dot{4}$, 12, $\dot{12}$, 9	$\dot{16}$, $\dot{64}$, 47, 8, 24
* 11	208, $\dot{38}$	$\dot{2}$	1, 5, 3, $\dot{7}$, 0	$\dot{5}$, $\dot{12}$, $\dot{2}$, 13, 8	$\dot{20}$, 6, $\dot{49}$, $\dot{4}$, 66
41	197, $\dot{23}$	2	$\dot{3}$, $\dot{1}$, $\dot{4}$, 2, 7	9, $\dot{6}$, 5, 1, $\dot{8}$	3, $\dot{20}$, $\dot{54}$, 62, 8
* 91	192, 10	0	$\dot{1}$, $\dot{2}$, 8, $\dot{2}$, $\dot{2}$	$a\omega$	23, $\dot{76}$, 14, $\dot{24}$, 14
9151	197, $\dot{21}$	$\dot{1}$	$\dot{2}$, 7, 0, 1, $\dot{5}$	1, $\dot{4}$, $\dot{7}$, 5, 7	$\dot{6}$, 56, $\dot{18}$, $\dot{57}$, 24
61	193, $\dot{11}$	1	$\dot{6}$, 1, $\dot{1}$, 6, 2	$\dot{2}$, 8, $\dot{6}$, 0, 1	50, $\dot{28}$, 3, $\dot{56}$, 30
81	207, $\dot{35}$	0	2, 7, $\dot{2}$, $\dot{1}$, $\dot{5}$	$a\omega$	2, 14, $\dot{30}$, $\dot{49}$, 62
9221	202, $\dot{28}$	2	8, 0, $\dot{3}$, $\dot{2}$, $\dot{2}$	$\dot{10}$, 5, 1, $\dot{3}$, 8	$\dot{74}$, 3, 14, 20, 36
* 41	213, 41	$\dot{1}$	$\dot{4}$, 7, 1, $\dot{4}$, 2	4, 3, 3, $\dot{2}$, $\dot{7}$	$\dot{12}$, $\dot{6}$, $\dot{61}$, 54, 24
81	193, 5	0	8, $\dot{2}$, 0, $\dot{3}$, $\dot{1}$	$a\omega$	72, $\dot{36}$, $\dot{10}$, $\dot{29}$, 2
9311	193, 1	$\dot{1}$	$\dot{1}$, $\dot{3}$, 8, 0, $\dot{2}$	$\dot{7}$, 3, 4, 5, $\dot{4}$	$\dot{40}$, 72, $\dot{2}$, $\dot{6}$, $\dot{25}$
41	197, $\dot{17}$	$\dot{2}$	2, 5, 0, 1, $\dot{7}$	$\dot{1}$, $\dot{11}$, $\dot{5}$, 6, 12	2, 30, $\dot{64}$, $\dot{13}$, 48
* 71	208, 34	1	$\dot{7}$, 3, 4, $\dot{1}$, 3	$\dot{5}$, 4, $\dot{1}$, $\dot{6}$, 9	66, 7, $\dot{48}$, 2, $\dot{28}$
91	203, 27	$\dot{2}$	3, $\dot{1}$, $\dot{6}$, 6, 0	$\dot{3}$, 7, 9, 4, $\dot{7}$	8, 50, $\dot{64}$, $\dot{18}$, 23
9421	197, $\dot{15}$	0	0, $\dot{5}$, 1, 7, $\dot{2}$	$a\omega$	56, 3, 24, $\dot{30}$, $\dot{54}$
* 31	212, 38	$\dot{2}$	$\dot{5}$, 2, 6, $\dot{4}$, 2	7, $\dot{4}$, $\dot{9}$, 1, 6	$\dot{33}$, $\dot{42}$, 68, $\dot{2}$, 8
61	217, $\dot{43}$	2	1, $\dot{7}$, 0, 6, 1	$\dot{7}$, 8, $\dot{1}$, $\dot{5}$, 6	35, 2, 48, $\dot{56}$, $\dot{30}$
91	197, $\dot{13}$	2	$\dot{3}$, $\dot{1}$, $\dot{4}$, 7, 2	4, $\dot{6}$, 10, $\dot{9}$, 2	$\dot{12}$, $\dot{10}$, 16, 62, $\dot{57}$
9511	207, $\dot{31}$	$\dot{1}$	5, $\dot{6}$, $\dot{3}$, 3, 2	0, 5, 7, $\dot{1}$, $\dot{9}$	$\dot{25}$, 12, 68, $\dot{46}$, $\dot{10}$
21	203, 25	0	4, $\dot{1}$, 6, $\dot{2}$, $\dot{5}$	$a\omega$	$\dot{9}$, 38, 34, $\dot{70}$, 6
51	197, 11	1	8, $\dot{1}$, $\dot{2}$, $\dot{1}$, $\dot{3}$	$\dot{4}$, 6, $\dot{2}$, $\dot{5}$, 7	$\dot{76}$, 6, 12, 23, 34
9601	197, 9	$\dot{1}$	3, $\dot{3}$, $\dot{5}$, 6, 0	1, 6, 3, 0, $\dot{8}$	$\dot{6}$, $\dot{34}$, 72, 3, $\dot{36}$
31	213, 37	$\dot{2}$	4, 5, $\dot{5}$, 2, $\dot{4}$	$\dot{7}$, $\dot{7}$, 5, 7, 4	12, $\dot{24}$, 15, 26, $\dot{78}$
* 61	207, 29	$\dot{1}$	0, $\dot{7}$, 5, 0, 3	5, 3, 3, $\dot{7}$, $\dot{2}$	20, $\dot{4}$, $\dot{9}$, $\dot{64}$, 56
9721	198, $\dot{8}$	$\dot{2}$	$\dot{5}$, 6, 1, 3, $\dot{3}$	9, $\dot{6}$, $\dot{9}$, $\dot{2}$, 10	36, $\dot{72}$, $\dot{6}$, 35, 6
81	198, 4	1	2, $\dot{1}$, 7, $\dot{1}$, $\dot{5}$	7, $\dot{8}$, 3, $\dot{1}$, 0	$\dot{33}$, 54, 30, $\dot{54}$, 2
91	212, 34	$\dot{1}$	$\dot{4}$, 0, $\dot{2}$, $\dot{1}$, 8	$\dot{3}$, 7, 4, $\dot{4}$, $\dot{2}$	8, 0, $\dot{74}$, 42, 23
9811	213, $\dot{35}$	0	$\dot{5}$, 5, 2, 4, $\dot{4}$	$a\omega$	30, $\dot{78}$, 8, 24, 15
51	203, 19	1	$\dot{5}$, 7, 0, 2, $\dot{2}$	$\dot{4}$, 6, $\dot{8}$, 3, 4	34, $\dot{64}$, $\dot{18}$, 48, $\dot{1}$
71	217, $\dot{39}$	1	7, 3, $\dot{3}$, $\dot{2}$, $\dot{4}$	$\dot{1}$, 4, 1, $\dot{7}$, 5	$\dot{54}$, $\dot{12}$, $\dot{16}$, 15, 66
* 9901	222, $\dot{44}$	1	$\dot{2}$, 8, 1, $\dot{4}$, $\dot{2}$	6, $\dot{1}$, 4, $\dot{1}$, $\dot{6}$	14, 15, 50, $\dot{70}$, $\dot{10}$
* 31	207, 25	0	7, 1, $\dot{4}$, 1, $\dot{4}$	$a\omega$	$\dot{58}$, $\dot{22}$, 8, 8, 63
* 41	203, 17	$\dot{2}$	$\dot{2}$, 2, $\dot{5}$, 0, 7	$\dot{3}$, 5, 10, 0, $\dot{10}$	$\dot{42}$, 34, 54, $\dot{46}$, 1
98801	653, 79	1	10, $\dot{3}$, $\dot{10}$, 7, $\dot{2}$	6, 1, $\dot{3}$, 13, $\dot{16}$	114, 131, $\dot{218}$, 8, $\dot{36}$
* 98911	637, $\dot{45}$	0	$\dot{12}$, $\dot{3}$, 7, 2, 7	$a\omega$	$\dot{180}$, $\dot{54}$, $\dot{9}$, 36, 206
98981	662, 92	2	$\dot{10}$, 2, $\dot{6}$, 5, 10	22, $\dot{18}$, 13, $\dot{1}$, $\dot{15}$	$\dot{88}$, $\dot{36}$, $\dot{120}$, 236, 7
1562051	2552, $\dot{230}$	0	$\dot{14}$, $\dot{4}$, $\dot{13}$, 8, 24	$a\omega$	$\dot{96}$, $\dot{209}$, $\dot{678}$, 848, 134
1671781	2587, 33	$\dot{2}$	5, 23, $\dot{2}$, $\dot{21}$, 6	$\dot{18}$, $\dot{38}$, 3, 39, 15	642, $\dot{6}$, 510, $\dot{634}$, 51

On Complex Primes formed with the Fifth Roots of Unity. (*Supplementary Note.*) *By* H. W. LLOYD TANNER. Received and read May 11th, 1893.

57.* The object of the present note is to investigate a method of determining whether a given complex integer is prime or composite, and, what is substantially the same question, whether the norm is prime or composite. The discussion is based upon a classification of complex integers according to their orders of complexity, as will now be explained. The product of four conjugate integers $a\omega$, $a\omega^2$, $a\omega^3$, $a\omega^4$ is a real number; but, for some integers, $a\omega$, the product of less than four of the conjugate integers is real. The least number of conjugate integers which gives a real product will be called the order of complexity—or simply the order—of each of the factors. It is easily seen that the number of factors in any real product of $a\omega$ and its conjugates must be a multiple of the order of $a\omega$. In particular, the order of any integer must be a divisor of 4; so that we have integers of the first, second, or fourth order.

58. In the above definition the meaning of "real" is tacitly extended. Numbers are called real if they can be made real (in the strict sense) by adding a suitable multiple of the complex zero. Otherwise, indeed, the statement that the norm of a complex integer is real would not be true. The following discussion is not complicated, and some difficulties are avoided by extending the meaning still more; and we shall speak of a complex integer as "real," or as equivalent to a real number r, if it is included in the formula

$$u\omega \,.\, r + j\Omega,$$

where $u\omega$ is a complex unit, Ω is the complex zero $(1+\omega+\omega^2+\omega^3+\omega^4)$ and j is a real integer. Now $u\omega$ is of the form $\pm(\omega+\omega^4)^i$, and does not contain a simple unit, ω^k, as a factor. For all the complex integers considered will be presumed to have the absolute terms fixed so as to satisfy Kummer's condition

$$a\omega \equiv a\,(1), \text{ mod}\,(1-\omega)^2.$$

Now if $a\omega$, $b\omega$ satisfy this condition, so will their sum and product; but multiplying by ω^k will violate the condition.

* The paragraphs of this note are numbered in continuation with those of the paper previously communicated.

Factors of the first order (Art. 59).

59. When $a\omega$ is a multiple of $b\omega$, we have

$$a\omega = b\omega \,.\, c\omega,$$

where $c\omega$ is a complex integer. If $b\omega$ is of the first order, then, r being a real integer,

$$b\omega = r \,.\, u\omega + j\Omega.$$

Hence

$$a\omega = (r \,.\, u\omega + j\Omega)\, c\omega$$
$$= r \,.\, c'\omega + j'\Omega,$$

where $c'\omega = u\omega \,.\, c\omega$ is another complex integer, and

$$j', = j\,(c_0 + c_1 + c_2 + c_3 + c_4),$$

is another real integer. Hence

$$a\omega - j' \,.\, \Omega = r \,.\, c'\omega,$$

and

$$a_0 \equiv a_1 \equiv a_2 \equiv a_3 \equiv a_4, \text{ mod } r.$$

The converse is also true; for evidently any factor common to all the coordinate-differences, $a_i - a_0$, is a factor of $a\omega - a_0 \,.\, \Omega$.

The norm $Na\omega$ is a multiple of r^4, since it is a diaphoric* function of the coordinates of $a\omega$ of the fourth degree.

It is not true that a factor r^4 of $Na\omega$ arises only from a real factor of $a\omega$.

Factors of the second order (Arts. 60–62).

60. When $b\omega$ is of the second order, a product $b\omega \,.\, b\omega^h$ is real. There are two sorts of integers of the second order; one for which $h = 4$, and one for which $h = 2$. If $h = 1$, the product $b\omega . b\omega$ is not real, unless $b\omega$ is of the first order; and the case in which $h = 3$ reduces to that in which $h = 2$; for, if $b\omega \,.\, b\omega^3$ is real, it is not altered when ω is changed to ω^2, so that $b\omega^2 \,.\, b\omega$ is real.

61. If $a\omega$ is a multiple of $b\omega$, an integer of the second order, such that $b\omega \,.\, b\omega^4$ is equivalent to a real number, say r, then $a\omega \,.\, a\omega^4$ will have a real factor r. But, using the notation of Art. 6,

$$a\omega \,.\, a\omega^4 = A_0 + A_1\,(\omega + \omega^4) + A_2\,(\omega^2 + \omega^3),$$

* Prof. Cayley, in his memoir on the Schwarzian derivative, uses this word to denote a function of the differences of specified arguments (*Camb. Phil. Trans.*, Vol. XIII., p. 12).

and, in order that this may contain the real factor r, it is necessary and sufficient that

$$A_0 \equiv A_1 \equiv A_2, \text{ mod } r.$$

In $Na\omega$ the factor r^2 will appear, for $Na\omega$ is a quadratic function of the differences of A_0, A_1, A_2.

It is to be noted that r is a factor common to X, Y, and so immediately detected. For

$$X = 2A_0 - A_1 - A_2 = (A_0 - A_1) + (A_0 - A_2),$$

and $$Y = \quad A_2 - A_1 = (A_0 - A_1) - (A_0 - A_2),$$

so that the G. C. M. of $A_0 - A_1$, $A_0 - A_2$ is also the G. C. M. of X, Y, or of $\frac{1}{2}X$, $\frac{1}{2}Y$, if X, Y are even.

Further, the factor r, determined as above, includes the squares of the factors of the first order, for $A_0 - A_1$, $A_0 - A_2$ are quadratic diaphoric functions of the coordinates of $a\omega$.

62. When $a\omega$ is a multiple of $b\omega$, an integer of the second order such that $b\omega \, . \, b\omega^2$ is equivalent to a real integer r, then $a\omega \, . \, a\omega^2$ contains the real factor r. Hence r is a factor of the coordinate-differences of the reciprocal factor

$$q\omega = -a\omega \, . \, a\omega^2 \quad \text{or} \quad a\omega \, . \, a\omega^2 + \Omega \quad \text{(Art. 28).}$$

The factors thus found include the squares of the factors of the first order, because the coordinate-differences of $q\omega$ are diaphoric functions of the coordinates of $a\omega$ (Art. 29), but they do not include the factors of the second order considered in Art. 61.

Examples (Art. 63).

63. The following examples will illustrate the theory of factors of the second order.

Example 1. $$a\omega = -2\omega + 2\omega^2 + \omega^3,$$

$$A_0 = 9, \quad A_1 = -2 = A_2,$$

$$A_0 - A_1 = A_0 - A_2 = 11, \qquad \text{G. C. M.} = 11;$$

$$X = 22, \quad Y = 0,$$

$$q\omega = 6 - 4\omega - 6\omega^3 + 3\omega^4, \qquad \text{G. C. M. of } q_i - q_2,$$

Hence $Na\omega$ is a multiple of 11^2, and if $b\omega$ represent a prime factor of 11, $a\omega$ is a multiple of $b\omega \,.\, b\omega^2$. It will be found that

$$b\omega = \omega+\omega^3-\omega^4,$$

and that there is no other factor.

Example 2. $a\omega = -4+\omega-2\omega^2+5\omega^3+2\omega^4$, G. C. M. (a_i-a_j), 1;

$$A_0 = 50, \quad A_1 = -14, \quad A_2 = -9,$$

$$A_0-A_1 = 64, \; A_0-A_2 = 59; \quad X = 123, \quad Y = 5, \quad \text{G. C. M.} = 1;$$

$$q\omega = -24+9\omega-2\omega^2+42\omega^3-24\omega^4,$$

$$q_i-q_0 = 33,\ 22,\ 66,\ 0, \qquad \text{G. C. M.} = 11;$$

$$Na\omega = 3751 = 31\,.\,11^2,$$

$$a\omega = b\omega \,.\, b\omega^{-1} c\omega,$$

where
$$b\omega = \omega+\omega^3-\omega^4, \quad Nb\omega = 11,$$
$$c\omega = 2\omega-\omega^2, \qquad Nc\omega = 31.$$

Example 3. $a\omega = 24-38\omega+14\omega^2+49\omega^3-48\omega^4$,

$$Na\omega = 42784681,$$

$$A_0 = 6921, \quad A_1 = -4262, \quad A_2 = 802,$$

$$\left.\begin{array}{c} A_0-A_2 = 6119, \quad A_2-A_1 = 5064 \\ X,\ Y = 17302,\ 5064, \end{array}\right\} \text{G.C.M.} = 211;$$

$$q\omega = 4842+68\omega-552\omega^2-1482\omega^3-2877\omega^4,$$

$$q_i-q_1 = 4774,\ 0,\ -620,\ -1550,\ -2945, \quad \text{G.C.M.} = 31.$$

Thus $Na\omega$ is a multiple of (and in fact equal to) $211^2 \times 31^2$,

$$a\omega = b\omega \,.\, b\omega^4 \,.\, c\omega \,.\, c\omega^2,$$

where
$$b\omega = 2\omega-\omega^2, \quad Nb\omega = 31,$$
and
$$c\omega = 3\omega-2\omega^4, \quad Nc\omega = 211.$$

Factors of the fourth order (Arts. 64–72).

64. The presence in $a\omega$ of two or more different factors of the fourth order, or of a power of such a factor, is indicated in two ways. Firstly, every integer point (X, Y) upon the hyperbola

$$X^2-5Y^2 = 4Na\omega,$$

which belongs to one such number, $a\omega$, belongs to at least one other

[redacted]

complex number, which cannot be derived from the first by adding complex zeros, multiplying into complex units, or changing ω to ω^k. Secondly, there are at least two integer points (X, Y) on the hyperbola whose coordinates satisfy the inequalities

$$0 < 5Y < X.$$

65. To verify these statements, let

$$Na\omega = Nb\omega \,.\, Nc\omega.$$

Consistently with this we may have either

$$a\omega = b\omega \,.\, c\omega, \quad \text{or} \quad a\omega = b\omega \,.\, c\omega^4,$$

or
$$a\omega = b\omega \,.\, c\omega^2, \quad \text{or} \quad a\omega = b\omega \,.\, c\omega^3;$$

moreover in any one of these equations $a\omega$ may be replaced by $a\omega^2$, $a\omega^3$, $a\omega^4$, without affecting the norm equation.

We take X, Y to be the integer point belonging to the product $b\omega \,.\, c\omega$, the square-sum and product-sums of which will be denoted by A_0, A_1, A_2, so that

$$X = 2A_0 - A_1 - A_2, \quad Y = A_2 - A_1, \quad X^2 - 5Y^2 = 4Nb\omega \,.\, Nc\omega = 4Na\omega.$$

If x, y, B_0, B_1, B_2 denote the corresponding quantities for $b\omega$, and ξ, η, C_0, C_1, C_2 those for $c\omega$, we have

$$x = 2B_0 - B_1 - B_2, \quad y = B_2 - B_1, \quad x^2 - 5y^2 = 4Nb\omega,$$

$$\xi = 2C_0 - C_1 - C_2, \quad \eta = C_2 - C_1, \quad \xi^2 - 5\eta^2 = 4Nc\omega.$$

By means of these expressions, and the relations

$$A_0 = B_0 C_0 + 2B_1 C_1 + 2B_2 C_2,$$

$$A_1 = B_0 C_1 + B_1 C_0 + B_1 C_2 + B_2 C_1 + B_2 C_2,$$

$$A_2 = B_0 C_2 + B_2 C_0 + B_2 C_1 + B_1 C_2 + B_1 C_1,$$

(Art. 8), it is easily verified that

$$2X = x\xi + 5y\eta, \quad 2Y = x\eta + y\xi.$$

First test (Arts. 66, 67).

66. Consider first the two products

$$b\omega \,.\, c\omega, \quad b\omega \,.\, c\omega^4,$$

which are forms of $a\omega$ consistent with the hypothesis that

$$Na\omega = Nb\omega \,.\, Nc\omega.$$

The square-sum and product-sums of $c\omega^4$ are identical with those of $c\omega$; and therefore the square-sum and product-sums of $b\omega . c\omega^4$ are identical with those of $b\omega . c\omega$, viz., they are A_0, A_1, A_2. Hence one and the same integer point (X, Y) belongs to these two products.

I say also that, with certain exceptions to be specified, these two complex numbers are distinct; that $b\omega . c\omega^4$ and its conjugates $b\omega^k . c\omega^{4k}$ are not included in the formula

$$\pm \epsilon^n . b\omega . c\omega + j\Omega \quad \text{........................(1)},$$

where ϵ is the cyclotomic unit $\omega + \omega^4$, Ω is the complex zero, and n, j are real integers. For suppose that $b\omega . b\omega^4$ is included in (1). Unless n vanishes the point which belongs to $\pm \epsilon^n b\omega . c\omega + j\Omega$ differs from (X, Y) which belongs to $b\omega . c\omega$ and to $b\omega . c\omega^4$ (Art. 11). Thus, if $b\omega . c\omega^4$ is included in the general form (1), we must have

$$b\omega . c\omega^4 = \pm b\omega . c\omega + j\Omega.$$

Now the coordinate-sums of $b\omega . c\omega$ and $b\omega . c\omega^4$ are the same, say s, and the last equation requires

$$s = \pm s + 5j.$$

Since s is not a multiple of 5, this gives $j = 0$, and

$$b\omega . c\omega = b\omega . c\omega^4,$$

so that $c\omega$ is a factor of the second order, giving $c\omega . c\omega^2$ equivalent to a real number which is a factor of the coordinate-differences of $q\omega$. But $c\omega$ is by hypothesis of the fourth order. The equivalence of $b\omega b\omega$, $b\omega b\omega^4$ is therefore impossible.

67. When $h = 4$, we have for the second complex number $b\omega^4 . c\omega$, giving the same case as that just discussed, with the *rôles* of $b\omega$, $c\omega$ interchanged. The cases of $h = 2$, $h = 3$ are similar to each other. The number $b\omega^2 . c\omega^3$ (assuming $h = 2$) belongs to the integer point $(X, -Y)$, since the product-sums (A_1, A_2) of $b\omega . c\omega^4$ are interchanged when ω is changed to ω^2. It is therefore to be examined whether in this case a value of n cannot be found which will make

$$\pm \epsilon^n . b\omega . c\omega + j\Omega = b\omega^2 . c\omega^3.$$

This would imply $\quad M^n(X, Y) = (X, -Y),$

where M means the matrix $\frac{1}{2}\begin{pmatrix} 3, & 5 \\ 1, & 3 \end{pmatrix}$. Now, if n were even, the vertex would be an integer point, and $4Na\omega$ an exact square. If n were odd,

the terminal points of the reduced segment (Q, Q in the figure of Art. 15) would be integer points, and $Na\omega/5$ would be the square of an integer. Except in these two cases there are two non-equivalent complex numbers $b\omega \,.\, c\omega$, $b\omega \,.\, c\omega^4$, belonging to one integer point (X, Y).

It is worth while to add that the same is true even when $c\omega$ is identical with $b\omega$; $b\omega^h \,.\, b\omega^{-h}$ is certainly distinct from $b\omega \,.\, b\omega$, for the former is of the second order, and the latter of the fourth. This shows that exceptions to our theorem cannot arise unless $Na\omega$ is a multiple of 5.

Second (Tchébicheff's) test (Arts. 68–72).

68. To verify the second theorem of Art. 64, we consider the two complex numbers $b\omega \,.\, c\omega$ and $b\omega \,.\, c\omega^2$. The points belonging to $c\omega$, $c\omega^2$ are (ξ, η) and $(\xi, -\eta)$ respectively; therefore the two points (X, Y) and (X', Y') which belong to $b\omega \,.\, c\omega$, $b\omega \,.\, c\omega^2$ are given (Art. 65) by the equations

$$2X = x\xi + 5y\eta, \quad 2Y = x\eta + y\xi,$$

$$2X' = x\xi - 5y\eta, \quad 2Y' = -x\eta + y\xi.$$

The two points are different unless $\eta = 0$, which implies that $Nc\omega$ is an exact square. But if η is not 0 the two points (X, Y), (X', Y') are not only different but belong to different ranges, so that in the reduced segment there will be two different points corresponding to the two complex numbers $b\omega \,.\, c\omega$ and $b\omega \,.\, c\omega^2$.

69. We have identically

$$2X', 2Y' = \begin{pmatrix} \lambda, & 5\mu \\ \mu, & \lambda \end{pmatrix}\!\begin{pmatrix} X, & Y \end{pmatrix},$$

where $$\frac{\lambda}{XX' - 5YY'} = \frac{\mu}{XY' - X'Y} = \frac{2}{X^2 - 5Y^2}.$$

Thus (X', Y') is or is not on the same range as (X, Y) according as

$$\tfrac{1}{2}\begin{pmatrix} \lambda, & 5\mu \\ \mu, & \lambda \end{pmatrix}\ldots, = \Lambda,$$

is or is not an integral power of

$$M, = \tfrac{1}{2}\begin{pmatrix} 3, & 5 \\ 1, & 3 \end{pmatrix}.$$

The necessary and sufficient condition for the former alternative is that λ, μ should be integers. In fact the terms of $2M^n$ are integers,

[redacted]

so that the condition is clearly necessary. To prove that it is sufficient, we note that

$$(XX'-5YY')^2-5(XY'-X'Y)^2 = (X^2-5Y^2)(X'^2-5Y'^2)$$
$$= (X^2-5Y^2)^2,$$

so that the values of λ, μ written above give

$$\lambda^2-5\mu^2 = 4.$$

But, (3, 1) being the minimum integer solution of this equation, every integral solution (λ, μ) is given by

$$\frac{\lambda+\mu\sqrt{5}}{2} = \left(\frac{3+\sqrt{5}}{2}\right)^n,$$

where n is integral, and this is tantamount to

$$\begin{pmatrix}\lambda/2, & 5\mu/2\\ \mu/2, & \lambda/2\end{pmatrix} = \begin{pmatrix}3/2, & 5/2\\ 1/2, & 3/2\end{pmatrix}^n.$$

For these two equations, considered as equations for λ, μ, are manifestly equivalent when $n = 1$, and in both cases a change of n into $n+1$ replaces 2λ, 2μ by $3\lambda+5\mu$, $\lambda+3\mu$.

70. By means of the formulæ of Arts. 68, 69, λ and μ can be expressed in terms of x, y, ξ, η, the result being

$$\lambda = 2(\xi^2+5\eta^2)/(\xi^2-5\eta^2), \quad \mu = -4\xi\eta/(\xi^2-5\eta^2).$$

Hence $$\lambda-2 = 20\eta^2/(\xi^2-5\eta^2),$$

an expression which is a positive proper fraction (or 0) if

$$20\eta^2 < \xi^2-5\eta^2;$$

but, as this inequality implies that $\xi \pm 5\eta$ are both positive, ξ being taken to be positive, it follows that, if (ξ, η) is a point in the reduced segment of its hyperbola, λ cannot be integral unless η is 0. Now we may always take $c\omega$ so that the corresponding integer point is in the reduced segment. For we have

$$b\omega \,.\, c\omega = \epsilon^{-n}\, b\omega \times \epsilon^n \,.\, c\omega,$$

a transformation by which neither the norms of the factors are altered, nor the product, and n may be taken so that $\epsilon^n . c\omega$ is "reduced." It follows then that unless $\eta = 0$, λ cannot be an integer, and accordingly (X', Y') is not on the range (X, Y).

The condition $$\eta = 0,$$
which has twice determined a case of exception, is equivalent to
$$C_1 = C_2$$
(Art. 65), so that, when $\eta = 0$,
$$c\omega \, . \, c\omega^{-1}, = C_0 + C_1 (\omega + \omega^4) + C_2 (\omega^2 + \omega^3),$$
is equivalent to a real number, $C_0 - C_1$, which is a factor common to ξ, η; and therefore to X, Y.

71. It will now be proved that (X', Y') is not a point of the range $(X, -Y)$. We have identically
$$2X', 2Y' = \begin{pmatrix} \lambda, & 5\mu \\ \mu, & \lambda \end{pmatrix} \begin{pmatrix} X, & -Y \end{pmatrix},$$
where
$$\frac{\lambda}{XX' + 5YY'} = \frac{\mu}{XY' + X'Y} = \frac{2}{X^2 - 5Y^2},$$
and hence, by the equations of Art. 68,
$$\lambda = 2\,(x^2 + 5y^2)/(x^2 - 5y^2), \quad \mu = 4xy/(x^2 - 5y^2).$$
Now these equations are similar to those obtained in Art. 70, and we infer, as in that article, that (X', Y') is not on the range $(X, -Y)$, unless $b\omega$ is a number of the second order, such that $b\omega \, . \, b\omega^{-1}$ is equivalent to a real number which is a common factor of x, y, and therefore of X, Y.

72. The foregoing considerations lead at once to the theorem announced. For the reduced point on the range (X', Y') cannot coincide with the reduced point on the range (X, Y), (Art. 70), nor with that which is on the range $(X, -Y)$. Hence the reduced point on the range (X', Y') has a different abscissa from that of the reduced point (X, Y). There are, therefore, four different reduced points, corresponding namely to $(X, \pm Y)$, $(X', \pm Y')$, two of which will have positive ordinates. There are, therefore, two different integer points (X, Y) such that
$$0 < 5Y < X.$$

73. In general, Tchébicheff's test is by far the more convenient. It is so, for instance, when the question is the primeness of a given real integer of the form $5n+1$ or of a complex integer $a\omega$. In the latter case, however, the results of Arts. 59, 61, 62 are useful in reducing calculations. But the first test has advantages when it is

required to distinguish primes from composite numbers in a set of complex numbers which have a common square-sum. Following the procedure described, Art. 53, we first obtain a set of coordinates. All factors of the first order are included in the G. C. M. of the differences of these coordinates. Then the values of the product-sums are determined, and the one which is numerically the less is recorded. If A_0-A_1, A_0-A_2 have no common factor but 1, the reciprocal factor $q\omega$ is formed, and if the G. C. M. of its coordinate-differences is 1, there are no factors of the second order. If, these tests being satisfied, there is no other entry of the same product-sum under any of the complex numbers in the set, then the complex number in question is prime. The omission of the A_0-A_1, A_0-A_2 test would not invalidate the conclusion, but the omission of the $q\omega$ test would, because the factors of the second order detected by $q\omega$ would not be detected by the final test (Art. 66).

The Singularities of the Optical Wave-Surface, Electric Stability, and Magnetic Rotatory Polarization. *By* J. Larmor. Read April 13th, 1893. Received, in extended form, July 14th, 1893.

Although Fresnel recognized that the two sheets of his optical wave-surface meet each other, he did not explicitly realize that they meet at conical points. This fact, and the striking phenomena of conical refraction which are dependent on it, was left for Sir W. R. Hamilton to discover. Yet it hardly needed very much discovery; for the roughest observation of the colours of crystalline plates in polarized light had shown that there are only two directions of single-ray velocity in a crystal, and therefore that the two sheets of the wave-surface meet each other (or at any rate come very close together) only along these directions—therefore not along any curve of intersection, but at definite points, which must be conical points on the surface. Thus, as Sir G. G. Stokes remarked long ago, the prediction of conical refraction does not constitute any demonstration of the exactness of Fresnel's wave-surface; though the two experiments of

Lloyd, if they could be made with an infinitely narrow beam, and with exactly parallel rays, would reveal the existence of actual singular tangent planes, and their normals the optic axes, as well as the existence of actual conical points.

The question suggests itself whether there is anything to prevent the different sheets of the wave-surface corresponding to any crystalline elastic medium from crossing each other. The hypothesis of their intersecting along a curve would not lead to any striking anomaly like conical refraction; but yet dynamical reasons can be readily assigned which forbid its occurrence.

We can realize in a general way that if in an optical wave-surface the velocities of the two types of transverse undulations corresponding to any plane front coincide, these types become merged together, and undulations can be propagated which correspond to any direction of vibration in the front; and we might hence infer that the front is a singular one, and not one of the infinite number of double tangent planes which could be drawn to two intersecting sheets of a wave-surface. But more precise and also more general reasons can be given.

In any homogeneous material structure in three dimensions of space, there are three velocities with which plane undulations can travel with a given direction of front. These velocities must be all real, for an imaginary value would imply dynamically a real exponential, with arbitrary sign, in the expression for the vibrations of the medium, and would therefore show that the medium could not subsist, owing to instability. These three velocities will usually be given by a cubic equation, in which the variable is the velocity squared.

If the medium is incompressible, one of the velocities is infinite, and the other pair are determined by a quadratic equation; this also applies in other cases which have been imagined in connexion with physical optics. If these two velocities are equal, the quadratic equation has equal roots, and therefore the expression under the radical sign in its solution must vanish. But the reality of the roots requires that this expression can never be negative, in whatever direction the wave-front may lie. It therefore only vanishes for a direction which makes it a minimum. Thus the wave-fronts of coincident velocities are confined to a definite number of planes which correspond to minimum values of this function, and of these only the ones are to be selected which make it equal to zero. In general then there are no such values, *i.e.*, there are no double tangent planes to the wave-surface to be anticipated; though it may happen excep-

tionally that there are certain isolated planes of this kind, which therefore touch the surface along a curve, like the singular planes of Fresnel's surface.

It is not possible at all for the two sheets of the wave-surface to have a continuous series of tangent planes; therefore its two sheets cannot intersect along a curve. The exceptional case of one singular tangent plane touching along a plane curve implies a depression in the sheet of which that curve is the rim; it is possible that at the bottom of this depression the two sheets may come together, thus forming a conical point, but for the reason given above they must not intersect along a curve.*

It is easy to conceive that a slight change in the constitution of the medium would just introduce imaginary terms into that velocity which corresponds to the direction of a singular tangent plane with a curve of contact round a nodal point; so that in this sense the constitution of the medium is labile. The abnormality of conical refraction thus indicates the immediate approach of instability.

These conclusions with respect to singularities on the wave-surface also hold good in the more general case, when all three velocities of propagation are finite. For the equality of a pair of roots in the cubic equation which determines the velocities implies equality of roots in the reducing quadratic which is introduced in the algebraical solution of the cubic. Now for the roots of the cubic to be real, the roots of this quadratic must be imaginary; therefore the expression under the radical sign in them must be negative. That is, the directions of equal wave-velocity are determined by the maximum values of this expression, subject to the condition that the maximum is in each case zero. The three sheets of the wave-surface therefore in general have no double tangent planes, and therefore no curve of intersection; but in special cases they may have isolated singular planes or conical points. Also, as before, in tracing the gradual change in the form of the wave-surface which corresponds to gradual change in the constitution of the medium, the final stage is arrived at when two sheets of the surface come into contact; any further change in that direction lands us in instability.

* It is interesting to compare this method of ascertaining the nature of the singular planes with the argument employed by Sir G. G. Stokes, "Report on Double Refraction," *Brit. Assoc. Report*, 1862, where he shows that if a depression exist on the wave-surface its rim must be a plane curve, from the consideration that, if it were not plane, then in certain directions four parallel tangent planes could be drawn to the surface, whereas physical reasons preclude the existence of four separate velocities of propagation.

It is, perhaps, not fanciful to see in the singularities of the optical wave-surface an indication that it is not derived from a purely elastic medium, but is modified or dominated by the inertia and the free periods of the molecules with which the pervading aether is loaded.* For although the influence of temperature and pressure produce the most marked alterations in the positions of the optic axes of a crystal, yet they do not destroy it as an optical medium.†

In the light of these considerations, the famous dynamical proposition of Green,‡ that it is possible so to choose the elastic constants of a crystalline medium that the displacement in two of its vibration-types shall be exactly transverse to the wave-front, and that this condition leads to Fresnel's expression for the velocities, but with opposite polarization, involves the rider that a medium so modified has been brought to the verge of instability; and this is so whatever be the velocity of its wave of normal displacement.

Electric Stability in Crystalline Media.

These general considerations may be illustrated and applied in a brief discussion of the question as to what relations must be assumed, between the electric and magnetic constants of a crystalline medium, in order that it may be stable as regards those electric vibrations in it which have been experimentally realized by Hertz, and which are supposed on strong grounds to involve the same mechanism as that by which light is propagated.

The fundamental equations of electrodynamics, on Maxwell's scheme, are the two circuital relations. The first of these relations (Ampère's) expresses that the circulation of the magnetic force round any circuit is equal to the electric flow through the aperture of that

* This view of the dynamics of refraction, which has been formulated in recent years with general approval, is, strange to say, the very first explanation that ever was offered on the Undulatory Theory. In the memoir by Thomas Young, "On the Theory of Light and Colours," *Phil. Trans.*, 1801, Prop. VII., the manner in which dispersion can be thus explained is very clearly expounded; but the author has gone wrong in the calculation of the mutual influence of the free periods of the aether and the matter molecules, partly owing to his usual manner of attempting to see through the phenomena without the aid of analysis. On this point a recollection of the analytical dynamics of Lagrange and Laplace, with which he was well acquainted, or even further reflection in his own manner, would have kept him right; his explanation would then have been in principle complete and unexceptionable, and might have directed mathematicians to more fruitful ground than the partial theories, based on simple heterogeneity of the medium, which have been in the meantime developed at great length by Cauchy and others.

† Thus, according to Kerr (*Phil. Mag.*, 1888), the effect of a tensile or compressile strain on glass is simply to convert it optically into a uniaxial crystal with its optic axis along the line of strain.

‡ Green, *Trans. Camb. Phil. Soc.*, 1839; *Collected Papers*, p. 292.

circuit, multiplied by 4π; the magnetic force is therefore derived from a potential only in those parts of the field which convey no current. The second relation (Faraday's) expresses that the circulation of the electric force round any circuit is equal to the time-rate of decrease of the flux of magnetic induction through the aperture of that circuit; the electric force is therefore derived from a potential only in regions in which the magnetic field is constant or null.

To obtain a complete scheme of equations for any given medium, there must be conjoined with these principles the experimental laws which connect in that medium the electric force and the magnetic force with their correlative fluxes, the electric current and the magnetic induction. When we utilize the facts that the convergence of each flux is null, while the circulation of each force is as above formulated, the specification of the electric relations of the medium will be complete.

To obtain analytical expressions, let (PQR) be the electric force, $(\alpha\beta\gamma)$ the magnetic force, (uvw) the electric current, and (abc) the magnetic induction, each specified by its components referred to rectangular axes; then the fundamental circuital relations of Maxwell's theory become

$$\left.\begin{aligned} 4\pi u &= \frac{d\gamma}{dy} - \frac{d\beta}{dz}, \\ 4\pi v &= \frac{d\alpha}{dz} - \frac{d\gamma}{dx}, \\ 4\pi w &= \frac{d\beta}{dx} - \frac{d\alpha}{dy}, \end{aligned}\right\} \qquad \left.\begin{aligned} -\frac{da}{dt} &= \frac{dR}{dy} - \frac{dQ}{dz}, \\ -\frac{db}{dt} &= \frac{dP}{dz} - \frac{dR}{dx}, \\ -\frac{dc}{dt} &= \frac{dQ}{dx} - \frac{dP}{dy}. \end{aligned}\right\}$$

If the medium is non-magnetic, or only so feebly magnetic that its magnetization may be neglected, we have

$$(a, b, c) = (\alpha, \beta, \gamma)$$

exactly. But if it is magnetic, we must, in order to proceed further, assume a coefficient of magnetic permeability μ, which will in the case of crystalline structure involve the six coefficients belonging to a self-conjugate linear system of equations.

If we confine ourselves throughout to dielectrics, the electric current is the time-rate of change of the electric induction (XYZ) divided by 4π.* For media of no specific electric inductive capacity,

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* It has been remarked by Heaviside that the equations would gain in symmetry by the adoption of a new unit of current, which would suppress this factor 4π.

or in which this capacity may be neglected, we have

$$(X, Y, Z) = (P, Q, R).$$

But for ponderable media we must assume a coefficient of electric permittivity K, which also will in the case of crystalline structure involve the six coefficients of a self-conjugate linear system of equations.

This restriction to self-conjugate equations is (under ordinary conditions) required in both the electric and magnetic relations, on the principles first applied by Lord Kelvin, to avoid the possibility of perpetual motions.*

Thus we have the two systems of equations

$$\left.\begin{aligned} X &= K_{11}P+K_{12}Q+K_{13}R, \\ Y &= K_{21}P+K_{22}Q+K_{23}R, \\ Z &= K_{31}P+K_{32}Q+K_{33}R, \end{aligned}\right\} \qquad \left.\begin{aligned} a &= \mu_{11}\alpha+\mu_{12}\beta+\mu_{13}\gamma, \\ b &= \mu_{21}\alpha+\mu_{22}\beta+\mu_{23}\gamma, \\ c &= \mu_{31}\alpha+\mu_{32}\beta+\mu_{33}\gamma, \end{aligned}\right\}$$

in each of which the conjugate coefficients, such as K_{12} and K_{21}, are equal; while also

$$4\pi\,(u, v, w) = \frac{d}{dt}\,(X, Y, Z).$$

In treating of the dielectric relations of crystals, it is usual to neglect their coefficient of magnetization in comparison with their coefficient of electric polarization. The obviously simplest course then is to choose axes of coordinates along the directions of principal electric polarization, so that the coefficients K_{12}, K_{13}, ... do not appear, and this choice of axes will still be suitable for a more detailed analysis, in which the effect of the feeble magnetization of the medium will have to be considered; so that then μ_{11}, μ_{22}, μ_{33} are each nearly unity, while μ_{12}, μ_{13}, ... are small.

It will be convenient to invert the form of the magnetic and electric equations, and write

$$\left.\begin{aligned} \alpha &= \mu_1'\,a+\mu_{12}'b+\mu_{13}'c, \\ \beta &= \mu_{21}'a+\mu_2'\,b+\mu_{23}'c, \\ \gamma &= \mu_{31}'a+\mu_{32}'b+\mu_3'\,c, \end{aligned}\right\} \qquad \left.\begin{aligned} P &= K_1'X, \\ Q &= K_2'Y, \\ R &= K_3'Z, \end{aligned}\right\}$$

where $$\mu_{12}' = \mu_{21}'.$$

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* Maxwell, *Electricity and Magnetism*, § 297.

Then, expressing the equations of the electric induction, we have

$$\frac{d^2X}{dt^2}=\frac{d}{dt}\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right)=\left(\mu'_{21}\frac{d}{dz}-\mu'_{31}\frac{d}{dy}\right)\left(K'_3\frac{dZ}{dy}-K'_2\frac{dY}{dz}\right)$$
$$+\left(\mu'_{22}\frac{d}{dz}-\mu'_{32}\frac{d}{dy}\right)\left(K'_1\frac{dX}{dz}-K'_3\frac{dZ}{dx}\right)$$
$$+\left(\mu'_{23}\frac{d}{dz}-\mu'_{33}\frac{d}{dy}\right)\left(K'_2\frac{dY}{dx}-K'_1\frac{dX}{dy}\right),$$

with two similar equations.

In particular, if the directions of the principal electric and magnetic axes coincide, μ'_{23}, ... are null, and

$$\frac{d^2X}{dt^2}=\mu'_{22}\frac{d}{dz}\left(K'_1\frac{dX}{dz}-K'_3\frac{dZ}{dx}\right)+\mu'_{33}\frac{d}{dy}\left(K'_1\frac{dX}{dy}-K'_2\frac{dY}{dx}\right).$$

These relations are reduced to a simpler form by writing (P, Q, R) for (K'_1X, K'_2Y, K'_3Z), that is by retaining the electric force as the variable.

In the special case of μ'_{23}, ... null,

$$\frac{1}{K'_1}\frac{d^2P}{dt^2}=\mu'_2\frac{d}{dz}\left(\frac{dP}{dz}-\frac{dR}{dx}\right)+\mu'_3\frac{d}{dy}\left(\frac{dP}{dy}-\frac{dQ}{dx}\right)$$
$$=\mu'_2\frac{d^2P}{dz^2}+\mu'_3\frac{d^2P}{dy^2}-\frac{d}{dx}\left(\mu'_2\frac{dR}{dz}+\mu'_3\frac{dQ}{dy}\right).$$

If μ'_1, μ'_2, μ'_3 are each unity, the type becomes

$$\frac{1}{K'_1}\frac{d^2P}{dt^2}=\nabla^2P-\frac{d}{dx}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right),$$

which leads in the well-known manner to Fresnel's laws of double refraction.

When the magnetic coefficients are different from unity, the same form of the equations is reached by making

$$(P', Q', R')=(\mu_1'^{-\frac{1}{2}}P,\ \mu_2'^{-\frac{1}{2}}Q,\ \mu_3'^{-\frac{1}{2}}R),$$
$$(x', y', z')=(\mu_1'^{-\frac{1}{2}}x,\ \mu_2'^{-\frac{1}{2}}y,\ \mu_3'^{-\frac{1}{2}}z);$$

so that

$$(K'_1\mu'_2\mu'_3)^{-1}\frac{d^2P'}{dt^2}=\left(\frac{d^2}{dx'^2}+\frac{d^2}{dy'^2}+\frac{d^2}{dz'^2}\right)P'-\frac{d}{dx'}\left(\frac{dP'}{dx'}+\frac{dQ'}{dy'}+\frac{dR'}{dz'}\right).$$

Thus the wave-surface in this case is of a form which may be derived from Fresnel's by application of homogeneous strain, according to the above specification.* Furthermore, the geometrical relations of

* Heaviside, *Phil. Mag.*, 1885; *Electrical Papers*, Vol. II., p. 16.

the electric force to the new wave-surface are correctly represented by imposing the same strain on its relations to the Fresnel surface from which the new one has been derived. The electric induction is in every case in the plane of the wave-front, on account of its solenoidal character.

For the most general case, Heaviside finds (*loc. cit.*), by the aid of a powerful vector analysis, that the equation for the velocities of the two waves whose fronts are in the direction (lmn) is

$$V^4 - \left[(K_2'\mu_{33}' + K_3'\mu_{22}')\,l^2 + \dots - 2K_3'\mu_{12}'\,lm - \dots\right] V^2$$
$$+ (\mu_{11}l^2 + \dots + 2\mu_{12}lm + \dots)(K_1 l^2 + \dots) = 0;$$

and that the equation of the wave-surface is

$$1 + [\mu]\,[K]\,(\mu_{11}'x^2 + \dots + 2\mu_{12}'xy + \dots)(K_1'x^2 + \dots)$$
$$- \left[(K_2\mu_{33} + K_3\mu_{22})\,x^2 + \dots - 2K_3xy - \dots\right]^2 = 0,$$

where $[\mu]$, $[K]$ are the determinants of the μ, K matrices.

The condition for the electric stability of the medium is that the two roots of this equation for V^2 should be real and positive for all values of the direction cosines l, m, n. This involves that the quartic cone

$$\left[(K_2'\mu_{33}' + K_3'\mu_{22}')\,x^2 + \dots\right]^2 = 4\,(\mu_{11}x^2 + \dots)(K_1x^2 + \dots)$$

should be wholly imaginary, or at most reduce to two rays. The directions of these two rays will in that special case be optic axes of the wave-surface, and will correspond to singular tangent planes which touch it all along a curve.

In the theory of crystalline refraction, based upon electric ideas, the difficulty is to some extent the opposite of that which occurs in purely dynamical theories; in the latter, the problem is so to combine the few independent relations allowed by the laws of pure dynamics as to represent all the phenomena; in the former each pair of related vectors are at the outset assumed to be connected in the most general linear manner, and we have to decide what is to be done with all the array of constants so introduced. The conditions of stability here indicated require relations between them.

It may, however, possibly be objected to this statement of the conditions of stability of the medium, that it is in discrepancy with the ordinary theory of opacity, which makes that quality depend on the existence of an imagi nary value for the velocity of propagation

But it is to be borne in mind that the terms which explain opacity, on the electromagnetic theory, involve electric conduction, and are therefore of a frictional character, and so irreversible; while, on the other hand, terms which depend on the structure of the medium are reversible, so that if a real exponential occur in the solution of the velocity equation with a negative sign, a reversal of the motion will make it appear with a positive sign, which would lead to a breaking-up of the medium.

The consideration of an imaginary index without accompanying instability also occurs in the theory of anomalous dispersion elaborated by Sellmeier, Helmholtz, and Kelvin; but there it enters from the consideration of the sympathetic vibrations of molecular structures which are considered to be *outside* the system which transmits the undulations, and which therefore extract its energy in much the same sort of way as frictional resistances. In fact a strong argument in favour of supposing that double refraction is to be explained in this manner, is that we thereby avoid the incipient instability which would be a characteristic of a simple elastic medium possessing that property.

Comparison of Theories of Dispersion and Rotatory Polarization.

A general formal development of the equations of the electromagnetic theory, which is necessarily wide enough to take account of all possible secondary phenomena, such as dispersion and circular polarization, was first given by Prof. Willard Gibbs,* under the title of "An Investigation of the Velocity of Plane Waves of Light, in which they are regarded as consisting of solenoidal electrical fluxes in an indefinitely extended medium of uniform and very fine-grained structure."

The principle on which his investigation is based is the very general idea that the regular simple harmonic light-waves traversing the medium excite secondary vibrations in its molecular electrical structure, which is supposed very fine compared with the length of a wave. When there is absorption, the phases of these excited vibrations will differ from that of the exciting wave; but even in this most general case the simple harmonic electric flux with which we are alone concerned is at each point completely specified by six

* In its final form this is contained in a paper, "On the General Equations of Monochromatic Light in Media of every Degree of Transparency," *American Journal of Science*, February, 1883.

quantities, the three components of the flux itself, and the three components of its rate of change with the time. In the same way, the electric force may be similarly specified by six quantities. Now the electric elasticity of the medium, as regards its power of transmitting waves, is specified by the relation connecting average force and average flux, this average referring to a region large compared with molecular structures, but small compared with a wave-length. The most general relation of this kind, that can result from the elimination of the molecular vibrations, must be of the form of six linear equations connecting the quantities specifying the flux with the quantities specifying the force, the coefficients being functions of the wave-length. If E denote the flux and U the force, "we may therefore write in vector notation

$$[E]_{\text{Ave}} = \Phi\,[U]_{\text{Ave}} + \Psi\,[\dot{U}]_{\text{Ave}},$$

where Φ and Ψ denote linear functions.*

"The optical properties of the media are determined by the forms of these functions. But all forms of linear functions would not be consistent with the principle of the conservation of energy.

"In media which are more or less opaque, and which therefore absorb energy, Ψ must be of such a form that the function always makes an acute angle (or none) with the independent variable. In perfectly transparent media Ψ must vanish, unless the function is at right angles to the independent variable. So far as is known, the last occurs only when the medium is subject to magnetic influence. In perfectly transparent media, the principle of the conservation of energy requires that Φ should be self-conjugate, *i.e.*, that for three directions at right angles to one another, the function and independent variable should coincide in direction.

"In all isotropic media not subject to magnetic influence, it is probable that Φ and Ψ reduce to numerical coefficients, as is certainly the case with Φ for transparent isotropic media."†

The subject of the rotatory polarization produced by a magnetic field has recently been resumed by various writers, with a view to the elucidation of the experimental results of Kerr, on the reflection of light from magnets. The following investigation of the degree of variety that it is permissible to import into the theoretical treatment

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* This requires that the coefficients K_{11}, K_{12}, ... above are complex quantities, depending in part on the period.

† J. Willard Gibbs, *loc. cit.*, p. 133.

may be of service in indicating the alternatives out of which a theory must be finally chosen; it will also exhibit the relation to each other of the various theories which have been developed.

To begin with, we recall the fundamental circuital relations of the types

$$4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz}, \quad \mu_1 \frac{da}{dt} = \frac{dR}{dy} - \frac{dQ}{dz},$$

in which the axes are those of principal magnetic permeability (μ_1, μ_2, μ_3). The relation between the current and the electric induction is, in a dielectric,

$$(u, v, w) = \frac{1}{4\pi} \frac{d}{dt}(X, Y, Z);$$

in a metallic or other conducting medium this would be replaced by

$$(u, v, w) = \frac{1}{4\pi} \frac{d}{dt}(X, Y, Z) + (\sigma)(P, Q, R).$$

The hypothesis adopted by Basset* and by Drude,† following FitzGerald and Rowland, is that the electric force contains, in addition to the part of it derived in Maxwell's manner from the kinetic (magnetic) energy of the medium, a part derived from a new term in the kinetic energy due to the indirect action of the imposed permanent magnetic field, giving the components

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ \dot{X} & \dot{Y} & \dot{Z} \end{vmatrix},$$

and that this is the only change introduced. The effect is simply that the magnetic force is to be derived not from the total electric force (P, Q, R), but from the part

$$(P', Q', R') \equiv (P + p_3\dot{Y} - p_2\dot{Z},\ Q + p_1\dot{Z} - p_3\dot{X},\ R + p_2\dot{X} - p_1\dot{Y});$$

and if we take (P', Q', R') as independent variables, the relation between $4\pi(u, v, w)$ [that is $(\dot{X}, \dot{Y}, \dot{Z})$] and (P', Q', R') will be expressed by a set of linear equations with rotational coefficients.

On the other hand, the hypothesis of Willard Gibbs‡ and Goldhammer§ takes the circuital relations as the fundamental and

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* *Phil. Trans.*, 1891; *Physical Optics*, Ch. xx.
† *Wied. Ann.*, XLVI., p. 377, 1892.
‡ *American Journal of Science*, 1883.
§ *Wied. Ann.*, 1892, XLVI., p. 75.

unmodifiable principles of electrodynamics, and assumes, after Maxwell, that the effect of the imposed magnetic field is to produce a rotational aeolotropy in the constitution of the medium, so that the system of linear equations connecting electric induction with electric force becomes rotational. This is clearly formally equivalent to the other process, if (P', Q', R') is there taken to be the electric force, *except* as to the mode in which the rotational coefficients involve the operator d/dt as a factor. Now, according to the considerations taken from Willard Gibbs, which have been sketched above, the most general possible type of his rotational coefficients, for simple harmonic vibrational disturbance, is $p+q\dfrac{d}{dt}$, where p and q are constants (vectorial) for the particular period involved; and Goldhammer makes the point that for slow periods, and for steady fields, this coefficient practically reduces to p, and includes the Hall effect, while for very rapid periods such as those of light-waves it reduces to $q\dfrac{d}{dt}$, which is the form required, as above, to make the two types of theory formally agree as to mode of propagation.* If the medium is non-conducting, the negation of a perpetual motion requires that the coefficients of type p shall form a self-conjugate system.

We now compare the boundary conditions that must be satisfied, on these two hypotheses, in a problem of reflexion. In ordinary electrodynamics, taking for the moment the axis of z normal to the interface, all the following quantities must be continuous across it,

$$P,\ Q,\ Z,$$
$$a,\ \beta,\ c.$$

Of these the continuity of a, β involves that of Z, by the first circuital relation, which is taken by Maxwell to be kinematic and not kinetic; and under ordinary circumstances the continuity of P, Q would involve that of c. But under the conditions of the first hypothesis, the continuity of P, Q violates that of c, which would rather require instead that P', Q' should be continuous. The solenoidal character of the electric and magnetic induction could hardly be subject to modification; we are driven therefore on the first hypothesis to admit that the electric force parallel to the interface is discontinuous in crossing it. The question whether both P' and Q' are continuous,

[* A discussion is also given by J. J. Thomson, *Recent Researches in Electricity and Magnetism*, 1893, p. 509.]

or only $\frac{dP'}{dy} - \frac{dQ'}{dx}$ is so, is usually settled by considering that the flow of energy across the interface is continuous, that is that there is no *quasi*-Peltier effect at the surface; taking Maxwell's expressions for the energy in terms of the electric and the magnetic induction, this necessitates the continuity of both P' and Q'.

On the hypothesis developed by Gibbs and Goldhammer, no difficulty of this kind arises. The boundary conditions are simply the ordinary ones; and they formally agree with the conditions finally assumed on the other hypothesis, for the reasons given above.

The general dynamical considerations which verify the restriction of the rotatory coefficient to Goldhammer's form have been briefly indicated at the end of a previous paper,* and may be recapitulated as follows. The equation of propagation of a plane wave is of type

$$\rho \frac{d^2\theta}{dt^2} = k \frac{d^2\theta}{dz^2}.$$

The introduction of small additional terms of odd order, and therefore of the types

$$\kappa_1 \frac{d^3\theta}{dz^2 dt}, \quad \kappa_2 \frac{d^3\theta}{dz^3}, \quad \kappa_3 \frac{d\theta}{dt}, \quad \kappa_4 \frac{d^3\theta}{dz\, dt^2}, \quad \kappa_5 \frac{d^3\theta}{dz^3},$$

will produce rotation of the plane of polarization. In the case of the first three types, change of sign of z does not affect the phenomenon; thus the rotation is in the same direction whether the wave travels forward or backward; it is of the magnetic kind. In the case of the fourth and fifth types, change of sign of z produces the same effect as change of sign of the rotatory coefficient; the rotation is of the kind exhibited by quartz and sugar and other active chemical compounds.

On an ultimate dynamical theory, $\rho \frac{d^2\theta}{dt^2}$ will represent kinetic energy; and the principle of dimensions shows that κ_1/ρ, κ_2/ρ, κ_3/ρ are respectively of dimensions $[L^2T^{-1}]$, $[T]$, $[T^{-1}]$. Thus the coefficient κ_1 will produce rotation owing to some influence of a distribution of angular momentum pervading the medium; while the coefficients κ_2 and κ_3 would produce selective rotation owing to the influence of the free periods of the fine-grained structure of the imbedded atoms of matter. The latter kind of rotation is to be expected only in the

* "The Equations of Propagation of Disturbances in Gyrostatically Loaded Media," *Proc. Math. Soc.*, 1891, p. 134.

rare cases in which selective absorption is prominent; consequently we are entitled, as a first approximation, to ascribe magnetic rotation to a coefficient of type κ_1; a conclusion which is abundantly verified by the discussion of the results of the experimental measures of Verdet and others.*

This review gives the conclusion that all effective theories of magnetic rotation, whether electro-optic, or gyrostatic and so purely dynamical, lead to the same modification of the equations of light-propagation, in order to take account of magnetic rotation *to a first approximation;* the consideration of the possible influence of ordinary and selective dispersion must in any case be conducted in a tentative manner. The different ways of formulating the electric theory also lead practically to the same boundary conditions in problems of reflexion at a magnet; while the consideration of what should be the boundary conditions on a gyrostatic theory† places in a prominent view what is the great trouble in the definite formulation of all problems of reflexion of light, viz., that a gradual transition at the boundary, taking place over a sensible portion of a wave-length, may completely alter the circumstances.

The result of a comparison of his formulæ with an elaborate series of measurements of Kerr's phenomenon, made by Sissingh,‡ leads Drude to the conclusion that all the features are fairly accounted for by a real coefficient of the type q; while a similar discussion by Goldhammer, a few months earlier, led to a more exact correspondence on taking q to be a complex quantity, as it is customary to assume the optical constants of metals to be. If, however, we are to found Goldhammer's equations on the physical basis afforded by the remarks of Willard Gibbs, quoted above, all the constants that occur must be real; and they must be able to account for metallic reflexion and all the other phenomena, so far as they are independent of dispersion.

The question whether a gradual transition at the reflecting surface will sensibly influence the modification imposed by magnetization on the reflexion has apparently not yet been examined.§

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* Maxwell, *Elec. and Mag.*, § 830.

† *Loc. cit.*, *Proc. Math. Soc.*, 1891.

‡ *Wied. Ann.*, 1891; trans. in *Phil. Mag.*, 1891.

[§ The making the coefficients q complex, by Goldhammer, involves a virtual reintroduction of the coefficients p, but with the essential difference that the new coefficients are combined with the frequency of vibration as a factor, and so have a preponderating influence when the frequency is very great.

In *Wied. Ann.*, 1893, Drude returns to this subject, and discusses some new

Effect of a Magnetic Field on Stability.

If the medium exhibits rotation of the plane of polarization in the Faraday manner, when it is placed in a powerful magnetic field parallel to the axis of z, the exact equations connecting electric induction with the electric force may thus be taken on Gibbs' theory to be

$$X = K_1 P - \epsilon \frac{d}{dt} Q,$$

$$Y = K_2 Q + \epsilon \frac{d}{dt} P,$$

$$Z = K_3 R,$$

where ϵ is a rotatory coefficient which is proportional to the strength of the magnetic field. It may here be noticed again that in a transparent medium these equations involve no absorption of energy.

To examine the effect of this rotational coefficient on the periods, it will suffice to take the medium as usual magnetically isotropic, or more simply non-magnetic, so that

$$(a, b, c) = (\alpha, \beta, \gamma).$$

The equations are in that case of the type

$$\frac{d^2 X}{dt^2} = \nabla^2 P - \frac{d}{dx}\left(\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz}\right).$$

From them, on substituting for (X, Y, Z) in terms of (P, Q, R), and writing

$$(P, Q, R) = (P_0, Q_0, R_0) \exp \iota \frac{2\pi}{\lambda} (lx + my + nz),$$

where P_0, Q_0, R_0 are functions of the time, there arises

$$\left(\frac{\lambda}{2\pi}\right)^2 \frac{d^2}{dt^2}\left(K_1 P_0 - \epsilon \frac{d}{dt} Q_0\right) = -P_0 + l\ (lP_0 + mQ_0 + nR_0),$$

$$\left(\frac{\lambda}{2\pi}\right)^2 \frac{d^2}{dt^2}\left(K_2 Q_0 + \epsilon \frac{d}{dt} P_0\right) = -Q_0 + m\ (lP_0 + mQ_0 + nR_0),$$

$$\left(\frac{\lambda}{2\pi}\right)^2 \frac{d^2}{dt^2} K_3 R_0 = -R_0 + n\ (lP_0 + mQ_0 + nR_0).$$

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observations of Zeeman on reflexion from cobalt, which were supposed to disagree with his formulæ. His conclusion seems to be that, although two coefficients naturally give a somewhat better account of the observations than one, yet the account they give is not complete, and suggests the residual influence of surface contamination or some other cause—that, in fact, the necessity for a complex constant cannot be allowed in the absence of more complete experimental data.]

The elimination of P_0, Q_0, R_0 yields an equation for the operator $\frac{d}{dt}$; calling this operator D, the equation is

$$\frac{K_1+\epsilon\frac{m}{l}D}{K_1+\frac{4\pi^2}{\lambda^2D^2}-\epsilon\frac{l}{m}D}+\frac{K_2-\epsilon\frac{l}{m}D}{K_2+\frac{4\pi^2}{\lambda^2D^2}+\epsilon\frac{m}{l}D}+\frac{K_3}{K_3+\frac{4\pi^2}{\lambda^2D^2}}$$

$$=\frac{lmn^2K_3\epsilon D\left\{\frac{K_1^2}{l^2}+\frac{K_2^2}{m^2}+\left(\frac{1}{l}+\frac{1}{m}\right)\frac{4\pi^2}{\lambda^2D^2}\right\}}{\{\ldots\ldots\}\{\ldots\ldots\}\{\ldots\ldots\}}.$$

This is an equation of the sixth degree, of which, when ϵ is small, the roots are nearly equal in pairs. One pair of them disappears when ϵ vanishes, so that they are very small, and correspond to waves which are propagated with extreme slowness. Thus, as Gibbs remarks, their wave-length would be much smaller than molecular magnitudes, and they therefore cannot have an actual existence.

Yet it would appear that, in strictness, without something which exactly fulfils the *rôle* of this wave, this theory of magnetic reflexion will come to grief through inability to run parallel to the actual physical conditions; though in the first approximation which coincides with the other theory these waves do not present themselves at all.

This electric theory runs in close correspondence with Lord Kelvin's theory of a labile mechanical aether, in which the velocity of compressional disturbances is null; the introduction of rotatory coefficients makes that velocity finite but small, exactly in the manner here exemplified. If we suppose this mechanical medium to be an adynamic gyrostatic structure, an aeolotropic arrangement of the axes of the gyrostats will represent crystalline quality, but there must be as many gyrostats with their axes pointing in one direction as there are with their axes pointing in the opposite direction; any cause which in addition slightly slews round the axes towards a certain direction will introduce rotation of the magnetic type with respec- to that direction.

We may now examine how far the rotatory terms or other small terms of the same order disturb the stability of a medium of biaxial character. The general analytical problem is as follows: along the direction of the optic axes the equation for V^2 has two equal roots; the coefficients in this equation are altered by the introduction of a series of new quantities of small magnitude typified by ϵ; it is

required to find the condition that the originally equal roots of the equation, thus altered, shall preserve real values. Let the value of a root of the equation

$$\phi(x) = 0,$$

where x stands for V^2, be thus altered from x to $x+\delta x$; then

$$\phi(x)+\phi'(x)\,\delta x+\tfrac{1}{2}\phi''(x)\,\delta x^2+\ldots+\Sigma\left\{\frac{d\phi}{d\epsilon}\,\delta\epsilon+\tfrac{1}{2}\,\frac{d^2\phi}{d\epsilon^2}\,\delta\epsilon^2+\ldots\right\}=0,$$

wherein $\phi(x) = 0,$

and also $\phi'(x) = 0,$

owing to the equality of roots. Thus

$$\tfrac{1}{2}\phi''(x)\,\delta x^2 = -\delta_\epsilon\phi,$$

where $\delta_\epsilon\phi$ represents the change in $\phi(x)$ produced by the new terms. In order that the roots may remain real, the alteration must be in such direction as to make the sign of $\delta_\epsilon\phi \Big/ \frac{d^2\phi}{dx^2}$ negative.

Now, as $\delta_\epsilon\phi$ is of the first degree in the increments $\delta\epsilon$, ..., it may have any sign at will; hence the medium thus modified is necessarily unstable. It must be concluded, if we adhere to a theory of this kind, that an imposed magnetic field will alter the electro-optic constants, not only by introducing rotatory coefficients, but also by modifying (very slightly) the double refraction of the medium so as to undo their tendency to instability.

The character of the circular polarization imposed magnetically on doubly-refracting media has been worked out by approximate methods by Prof. Willard Gibbs;* he has not, however, noticed that along the optic axes his approximation becomes nugatory, and that he has really to deal with conditions involving instability.

The theorem that the stability depends on the sign of $\delta_\epsilon\phi$ may be applied to obtain explicitly the conditions of stability of a slightly magnetic biaxial medium from Heaviside's formula (quoted *supra*) for the velocities; but the result is too long to merit reproduction.

In the case of the natural asymmetry of quartz, a uniaxial crystal, we can make a distinction between the ordinary and the extraordinary wave; and the consideration of stability gives a necessary reason for the relation derived by Airy from an experimental examination, that the wave-surface is so modified by rotational quality that one sheet lies wholly inside the other, instead of intersecting it along a curve as would be formally possible. A similar statement of course applies to the temporary modification produced by a strong magnetic field.

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* *American Journal of Science*, June, 1882.

To examine further the form these anomalies assume in the more simple media, let us suppose $K_1 = K_2$ ($= K$, say), so that the medium is uniaxial, with its axis along the direction of the magnetic field. The result will now be pure circular polarization, so that we are prompted to replace in the analysis

$$X,\ Y \ \text{by}\ \Xi = X+\iota Y,\quad \mathrm{H} = X-\iota Y,$$
$$P,\ Q \ \text{by}\ \Pi = P+\iota Q,\quad \Sigma = P-\iota Q,$$

and to take as coordinates

$$\xi = x+\iota y,\quad \eta = x-\iota y.$$

This leads to
$$\Xi = \left(K+\iota\epsilon\frac{d}{dt}\right)\Pi,$$
$$\mathrm{H} = \left(K-\iota\epsilon\frac{d}{dt}\right)\Sigma,$$

and
$$\frac{d^2\Xi}{dt^2} = \nabla^2\Pi - \frac{d}{d\xi}\left(\frac{d\Pi}{d\xi}+\frac{d\Sigma}{d\eta}+\frac{dR}{dz}\right),$$
$$\frac{d^2\mathrm{H}}{dt^2} = \nabla^2\Sigma - \frac{d}{d\eta}\left(\frac{d\Pi}{d\xi}+\frac{d\Sigma}{d\eta}+\frac{dR}{dz}\right),$$
$$\frac{d^2Z}{dt^2} = \nabla^2R - \frac{d}{dz}\left(\frac{d\Pi}{d\xi}+\frac{d\Sigma}{d\eta}+\frac{dR}{dz}\right).$$

Thus, on writing
$$(\Xi,\ \mathrm{H},\ R) = (\Xi_0,\ H_0,\ R_0)\exp\iota\frac{2\pi}{\lambda}(l\xi+m\eta+nz-Vt),$$

these equations are still formally the same as Fresnel's equations for biaxial media, when the reciprocals of the squares of the principal velocities are taken to be

$$K+\frac{2\pi\epsilon}{\lambda}V,\quad K-\frac{2\pi\epsilon}{\lambda}V,\quad K_3;$$

so that the velocities corresponding to the direction l, m, n are given by the equation

$$\frac{l^2}{\left(K+\frac{2\pi\epsilon}{\lambda}V\right)^{-1}-V^2}+\frac{m^2}{\left(K-\frac{2\pi\epsilon}{\lambda}V\right)^{-1}-V^2}+\frac{n^2}{K_3^{-1}-V^2}=0,$$

which has still three pairs of nearly equal roots, one pair of them small, when ϵ is small.

The equation for a medium originally isotropic is obtained on writing K instead of K_3; when expanded, it assumes the form

$$(1-KV^2)\left\{K+\frac{2\pi\epsilon}{\lambda}(l^2-m^2)KV^3-\left(\frac{2\pi\epsilon}{\lambda}\right)^2(l^2+m^2)V^4\right\}$$
$$+\left(\frac{2\pi\epsilon}{\lambda}\right)^2 n^2KV^6 = 0.$$

There are still three pairs of roots, of which two pairs are now very small when ϵ is very small; and, inasmuch as when ϵ is null these pairs of roots are equal, being zero, they may in the actual case become imaginary for certain values of l, m, n. Thus the instability here attaches to the very slow waves which are outside the limits of physical reality. With a rotatory coefficient of the p type, the trouble would not occur at all.

On the Linear Transformations between Two Quadrics. By HENRY TABER, Clark University, Mass., U.S.A. Received May 6th, 1893. Read May 11th, 1893.

1. *Introductory.*

In *Crelle's Journal*, Vol. L. (also *Phil. Trans.*, 1858), Cayley gave a representation of the automorphic linear transformation of the unipartite quadric function in the notation of the theory of matrices. In this paper I extend Cayley's method to the determination of the general linear transformation of a given quadric into another given quadric, and apply the results to the determination of the general real linear transformation between two equivalent quadrics and to the reduction of a quadric to a sum of squares. The determination by this method of the general linear transformation between two quadrics depends upon the solution of an algebraic equation of the n^{th} degree; to which the problem, as it originally presents itself, namely, the solution of a system of n^2 quadratic equations in n^2 variables, is thus reducible.

Following Cayley, the matrix of a linear transformation will, in this paper, be regarded as a quantity susceptible of addition. The *sum* or *difference* of two matrices is defined as follows:—

$$\left(\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{array}\right) \pm \left(\begin{array}{cccc} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{array}\right) = \left(\begin{array}{cccc} a_{11}\pm b_{11} & a_{12}\pm b_{12} & \dots & a_{1n}\pm b_{1n} \\ a_{21}\pm b_{21} & a_{22}\pm b_{22} & \dots & a_{2n}\pm b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}\pm b_{n1} & a_{n2}\pm b_{n2} & \dots & a_{nn}\pm b_{nn} \end{array}\right).$$

Products of matrices will be taken as equivalent to compound substitutions. Multiplication is then distributive over addition, *i.e.*,

$$\phi(\psi+\chi) = \phi\psi+\phi\chi,$$
$$(\phi+\psi)\chi = \phi\chi+\psi\chi,$$

for any three matrices ϕ, ψ, and χ. A *scalar** m, regarded as a matrix, has the representation

$$\begin{pmatrix} m & 0 & \dots & 0 \\ 0 & m & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & m \end{pmatrix}.$$

An equality between two matrices implies the equality of its corresponding constituents.

If ϕ is the matrix

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

the scalars g, for which

$$\begin{vmatrix} a_{11}-g & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-g & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-g \end{vmatrix} = 0,$$

will, following Sylvester, be termed the *latent roots* of ϕ. To denote the determinant of ϕ, the notation $|\phi|$ will be employed; and, since the left-hand member of the above equation is the determinant of $\phi-g$, it will be denoted by $|\phi-g|$.

If the determinant of ϕ is not zero, and if

$$\Phi = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix},$$

where A_{rs} denotes the first minor of $|\phi|$ with respect to a_{rs}, divided by $|\phi|$, then

$$\phi\Phi = \Phi\phi = 1;$$

i.e., Φ is the *reciprocal* of ϕ. We may denote Φ by ϕ^{-1}. We have

$$(\phi\psi)^{-1} = \psi^{-1}\phi^{-1}.$$

* I employ the term *scalar*, as Hamilton has done, to designate the quantities real and imaginary of ordinary algebra, in order to distinguish these from matrices regarded as quantities; the latter are *non-scalar* quantities.

The *transverse* of ϕ (*i.e.*, the matrix obtained by interchanging the rows and columns of ϕ) will be denoted by $\breve{\phi}$. We have

$$\breve{\breve{(\phi)}} = \phi,$$

$$\overbrace{(\phi+\psi)} = \breve{\phi}+\breve{\psi},$$

$$\breve{(\phi\psi)} = \breve{\psi}\breve{\phi},$$

$$\breve{(\phi^{-1})} = (\breve{\phi})^{-1}.$$

The matrix ϕ is *symmetric* if $a_{rs} = a_{sr}$ for $r, s = 1, 2, \dots n$, where a_{rs} denotes the constituent of ϕ in the r^{th} row and s^{th} column. The necessary and sufficient condition for a symmetric matrix is

$$\breve{\phi} = \phi.$$

The matrix ϕ is skew-symmetric if $a_{rs} = -a_{sr}$ for $r, s = 1, 2, \dots n$; for which the necessary and sufficient condition is

$$\breve{\phi} = -\phi.$$

If μ is any positive integer, I shall take $\phi^{1/\mu}$ to denote the matrix whose μ^{th} power is equal to ϕ. The matrix $\phi^{1/\mu}$ may be termed the μ_{th} root of ϕ. Every matrix whose determinant does not vanish has a μ^{th} root for any index μ. In general, if the order of the matrix is n, the number of the μ^{th} roots is μ^n. In the *Comptes Rendus*, XCIV., Sylvester gave the following expression for the μ^{th} root of any matrix ϕ whose latent roots $g_1, g_2, \dots g_n$, are all distinct, viz.,

$$\sum_1^n{}_r\, g_r^{1/\mu} \frac{(\phi-g_1)(\phi-g_2)\dots(\phi-g_{r-1})(\phi-g_{r+1})\dots(\phi-g_n)}{(g_r-g_1)(g_r-g_2)\dots(g_r-g_{r-1})(g_r-g_{r+1})\dots(g_r-g_n)}.$$

When two or more of the latent roots of ϕ are equal, the expression for $\phi^{1/\mu}$, as stated by Sylvester, may be obtained by expanding this expression in powers of ϕ, and finding the limiting values of the coefficients. If ϕ is real and symmetric, for the case of equal latent roots, $\phi^{1/\mu}$ may be obtained simply by taking the summation for the distinct latent roots.* It is to be observed that the roots of a symmetric matrix given by Sylvester's formula (as they are linear in powers of a symmetric matrix) are themselves symmetric.

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* See *Proc. Lond. Math. Soc.*, Vol. XXII., p. 461. The same is true for any matrix ϕ, for which the nullity of $|\,\phi-g\,|$, for any multiple latent root g, is equal to the multiplicity of g. If ϕ is of order n, nullity m is equivalent to *rank* (*Rang*) $n-m$.

We may obtain expressions for $(\phi^{-1})^{1/\mu}$ by substituting, in the above summation, $g_r^{-1/\mu}$ for $g_r^{1/\mu}$.* We have

$$(\phi^{-1})^{1/\mu} = (\phi^{1/\mu})^{-1};$$

either of these expressions may, therefore, be denoted simply by $\phi^{-1/\mu}$.

2. *Cayley's expression for the automorphic linear transformation of a quadric.*

Let Ω denote the matrix of the coefficients of the quadric

$$\sum_1^n a_{rs}\, x_r\, x_s,$$

where
$$a_{rs} = a_{sr}.$$

If the variables x are transformed by the linear substitution or matrix ϕ, so that

$$x_r = \sum_1^n {}_i\, \phi_{ri}\, \xi_i,$$

the matrix of the quadric so obtained will be equal to

$$\breve{\phi}\, \Omega\, \phi.\dagger$$

It may be required that the quadric so obtained shall be identically equal to the quadric $\Sigma b_{rs} \xi_r \xi_s$ (in which $b_{rs} = b_{sr}$), the matrix of whose coefficients is Ω'; for this the necessary and sufficient condition is that ϕ shall satisfy the matrical equation

$$\breve{\phi}\, \Omega\, \phi = \Omega'.$$

It will be assumed that the determinants of the matrices Ω, Ω', and ϕ, do not either of them vanish. Since, by supposition,

$$a_{rs} = a_{sr}, \quad \text{and} \quad b_{rs} = b_{sr},$$

Ω and Ω' are symmetric matrices.

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* This follows from the following more general formula for any function of the matrix ϕ, given by Sylvester in the *Johns Hopkins University Circulars*, Vol. III., viz.,

$$f(\phi) = \sum_1^n {}_r f(g_r) \frac{(\phi-g_1)(\phi-g_2)\dots(\phi-g_{r-1})(\phi-g_{r+1})\dots(\phi-g_n)}{(g_r-g_1)(g_r-g_2)\dots(g_r-g_{r-1})(g_r-g_{r+1})\dots(g_r-g_n)}.$$

† Cayley, "Memoir on the Automorphic Linear Transformation of the Bipartite Quadric Function," *Phil. Trans.*, 1858.

For the case of automorphic transformation, *i.e.*, when $\Omega = \Omega'$, Cayley has given the following expression for ϕ, viz.,

$$\phi = \Omega^{-1}(\Omega+\Upsilon)(\Omega-\Upsilon)^{-1}\Omega,$$

in which Υ is an arbitrary skew-symmetric matrix. From this expression for ϕ, it follows that

$$\begin{aligned}\breve{\phi} &= \breve{\Omega}\,(\breve{\Omega}-\breve{\Upsilon})^{-1}\,(\breve{\Omega}+\breve{\Upsilon})\,\breve{\Omega}^{-1}\\ &= \Omega\,(\Omega+\Upsilon)^{-1}\,(\Omega-\Upsilon)\,\Omega^{-1};\end{aligned}$$

therefore, substituting these expressions in the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega,$$

we should have

$$\Omega\,(\Omega+\Upsilon)^{-1}\,(\Omega-\Upsilon)\,\Omega^{-1}.\Omega.\Omega^{-1}\,(\Omega+\Upsilon)(\Omega-\Upsilon)^{-1}\,\Omega = \Omega.$$

But $(\Omega-\Upsilon)\,\Omega^{-1}\,(\Omega+\Upsilon) = \Omega-\Upsilon+\Upsilon-\Upsilon\Omega^{-1}\Upsilon = (\Omega+\Upsilon)\,\Omega^{-1}\,(\Omega-\Upsilon)$;

from which follow, successively,

$$(\Omega+\Upsilon)^{-1}\,(\Omega-\Upsilon)\,\Omega^{-1}\,(\Omega+\Upsilon)(\Omega-\Upsilon)^{-1} = \Omega^{-1},$$

$$\Omega\,(\Omega+\Upsilon)^{-1}\,(\Omega-\Upsilon)\,\Omega^{-1}.\,\Omega.\Omega^{-1}\,(\Omega+\Upsilon)(\Omega-\Upsilon)^{-1}\,\Omega = \Omega.^{*}$$

Equating the reciprocals of either side of the last equation but one gives

$$(\Omega-\Upsilon)(\Omega+\Upsilon)^{-1}.\Omega.(\Omega-\Upsilon)^{-1}\,(\Omega+\Upsilon) = \Omega;$$

but, if $\qquad \phi = (\Omega-\Upsilon)^{-1}\,(\Omega+\Upsilon),$

then

$$\begin{aligned}\breve{\phi} &= (\breve{\Omega}+\breve{\Upsilon})(\breve{\Omega}-\breve{\Upsilon})^{-1}\\ &= (\Omega-\Upsilon)(\Omega+\Upsilon)^{-1};\end{aligned}$$

therefore this expression for ϕ is also a solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega.$$

This expression is, however, identically equal to Cayley's expression; for from it we derive

$$\breve{\phi}^{-1} = (\Omega+\Upsilon)(\Omega-\Upsilon)^{-1};$$

and from the equation $\qquad \breve{\phi}\,\Omega\,\phi = \Omega$

* Cayley, *ibid.*

it follows that $$\phi = \Omega^{-1}\breve{\phi}^{-1}\Omega;$$
therefore $$(\Omega-\Upsilon)^{-1}(\Omega+\Upsilon) = \Omega^{-1}(\Omega+\Upsilon)(\Omega-\Upsilon)^{-1}\Omega.*$$
As the expression $(\Omega-\Upsilon)^{-1}(\Omega+\Upsilon)$ is somewhat simpler than its equivalent, I shall employ it in what follows.

Cayley's solution fails if it is required that ϕ shall have -1 as a latent root, but not otherwise. For, if ϕ is any solution of the equation
$$\breve{\phi}\,\Omega\,\phi = \Omega,$$
provided -1 is not a latent root of ϕ, we may put
$$\Upsilon = \Omega\,(\phi-1)(\phi+1)^{-1},$$
whence we obtain $$\breve{\Upsilon} = -\Upsilon,$$
since $$\breve{\phi} = \Omega\phi^{-1}\Omega^{-1},\dagger$$
and also $$\phi = (\Omega-\Upsilon)^{-1}(\Omega+\Upsilon);$$
but, since
$$\begin{aligned}\Omega-\Upsilon &= \Omega\left[1-(\phi-1)(\phi+1)^{-1}\right]\\ &= \Omega\,(\overline{\phi+1}-\overline{\phi-1})(\phi+1)^{-1}\\ &= 2\Omega\,(\phi+1)^{-1},\end{aligned}$$

---

* The identity between the two expressions may also be shown as follows:—
$$\begin{aligned}\Omega^{-1}(\Omega+\Upsilon)(\Omega-\Upsilon)^{-1}\Omega &= \Omega^{-1}.\,\Omega^{\frac12}(1+\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})\,\Omega^{\frac12}.\,\Omega^{-\frac12}(1-\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})^{-1}\Omega^{-\frac12}.\,\Omega\\ &= \Omega^{-\frac12}(1+\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})(1-\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})^{-1}\Omega^{\frac12}\\ &= \Omega^{-\frac12}(1-\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})^{-1}(1+\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})\,\Omega^{\frac12}\\ &= \Omega^{-\frac12}(1-\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})^{-1}\Omega^{-\frac12}.\,\Omega^{\frac12}(1+\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})\,\Omega^{\frac12}\\ &= (\Omega-\Upsilon)^{-1}(\Omega+\Upsilon).\end{aligned}$$

† For, if $$\breve{\phi}\,\Omega\,\phi = \Omega,$$
we have $$\breve{\phi} = \Omega\,\phi^{-1}\Omega^{-1};$$
therefore, if $$\Upsilon = \Omega\,(\phi-1)(\phi+1)^{-1},$$
$$\begin{aligned}\breve{\Upsilon} &= (\breve{\phi}+1)^{-1}(\breve{\phi}-1)\,\breve{\Omega}\\ &= (\Omega\phi^{-1}\Omega^{-1}+1)^{-1}(\Omega\phi^{-1}\Omega^{-1}-1)\,\Omega\\ &= \left[\Omega\,(\phi^{-1}+1)\,\Omega^{-1}\right]^{-1}.\,\Omega\,(\phi^{-1}-1)\,\Omega^{-1}.\,\Omega\\ &= \Omega\,(\phi^{-1}+1)^{-1}\Omega^{-1}.\,\Omega\,(\phi^{-1}-1)\,\Omega^{-1}.\,\Omega\\ &= \Omega\,(\phi^{-1}+1)^{-1}(\phi^{-1}-1)\\ &= \Omega\left[\phi^{-1}(1+\phi)\right]^{-1}.\,\phi^{-1}(1-\phi)\\ &= \Omega\,(1+\phi)^{-1}\phi.\,\phi^{-1}(1-\phi)\\ &= -\Omega\,(\phi+1)^{-1}(\phi-1).\end{aligned}$$

it is evident that $\Omega-\Upsilon$ has a reciprocal, and consequently the above expression for ϕ in terms of the skew-symmetric matrix Υ is possible. If, however, -1 is a latent root of ϕ, then $\phi+1$ has no reciprocal, and we cannot put

$$\Upsilon = \Omega(\phi-1)(\phi+1)^{-1},$$

which is required by Cayley's expression for ϕ in terms of Υ.*

If -1 is a latent root of ϕ, but not $+1$, then $+1$ will be a latent root of $-\phi$, but not -1; for, if

$$|\phi+1| = 0, \quad |\phi-1| \neq 0,$$

then

$$|(-\phi)-1| = 0, \quad |(-\phi)+1| \neq 0.$$

Therefore $-\phi$ can be represented as above, giving

$$\phi = -(\Omega-\Upsilon)^{-1}(\Omega+\Upsilon).$$

Thus the expression $\pm(\Omega-\Upsilon)^{-1}(\Omega+\Upsilon)$

gives every solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega,$$

except those for which both ± 1 are latent roots.

3. *Determination of the linear transformations of one quadric into another.*

Cayley's solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega$$

gives at once the means of solving the more general equation

$$\phi\,\Omega\,\phi = \Omega',$$

where Ω and Ω' are known symmetric matrices. For this equation may be written

$$\breve{\phi}\,\Omega^{\frac{1}{2}}\Omega'^{-\frac{1}{2}}\,.\,\Omega'\,.\,\Omega'^{-\frac{1}{2}}\Omega^{\frac{1}{2}}\phi = \Omega';$$

---

* Moreover, if $\phi = (\Omega-\Upsilon)^{-1}(\Omega+\Upsilon)$,
and if -1 is a latent root of ϕ, then

$$\frac{2^n}{|\Omega-\Upsilon|}|\Omega| = |(\Omega-\Upsilon)^{-1}(\overline{\Omega+\Upsilon}+\overline{\Omega-\Upsilon})|$$
$$= |(\Omega-\Upsilon)^{-1}(\Omega+\Upsilon)+1| = |\phi+1| = 0,$$

which is impossible, since, by supposition, $|\Omega| \neq 0$.

and, if $\Omega^{\frac{1}{2}}$ and $\Omega'^{-\frac{1}{2}}$ denote symmetric square roots of Ω and Ω'^{-1}, respectively, and

$$\psi = \Omega'^{-\frac{1}{2}}\,\Omega^{\frac{1}{2}}\,\phi,$$

the equation becomes
$$\breve{\psi}\,\Omega'\,\psi = \Omega',$$

of which the general solution is

$$\psi = \pm\,(\Omega'-\Upsilon)^{-1}\,(\Omega'+\Upsilon),$$

where Υ is an arbitrary skew-symmetric matrix. Therefore, *the general expression for the matrix ϕ, satisfying the equation*

$$\breve{\phi}\,\Omega\,\phi = \Omega'$$

(*i.e., the general expression for the linear substitution that will transform a given quadric whose matrix is Ω into another given quadric whose matrix is Ω'*), *is*

$$\pm\,\Omega^{-\frac{1}{2}}\Omega'^{\frac{1}{2}}\,(\Omega'-\Upsilon)^{-1}\,(\Omega'+\Upsilon),$$

where Υ is an arbitrary skew-symmetric matrix, and $\Omega^{-\frac{1}{2}}$ and $\Omega'^{\frac{1}{2}}$ are symmetric square roots of Ω^{-1} and Ω' respectively. Expressions for $\Omega^{-\frac{1}{2}}$ and $\Omega'^{\frac{1}{2}}$ may be obtained by means of Sylvester's formula.

This solution fails if the condition is imposed that

$$\psi = \Omega'^{-\frac{1}{2}}\,\Omega^{\frac{1}{2}}\,\phi$$

shall have as latent roots both ± 1. If $\left\{\begin{matrix}+1\\-1\end{matrix}\right\}$ is a latent root of ψ, but not $\left\{\begin{matrix}-1\\+1\end{matrix}\right\}$, the $\left\{\begin{matrix}\text{upper}\\\text{lower}\end{matrix}\right\}$ sign is to be taken.

Another form of the general solution is

$$\phi = \pm\,(\Omega-\Upsilon)^{-1}\,(\Omega+\Upsilon)\,\Omega^{-\frac{1}{2}}\,\Omega'^{\frac{1}{2}},$$

where Υ is an arbitrary skew-symmetric matrix. This solution may be verified by substituting for ϕ in $\breve{\phi}\,\Omega\,\phi$, giving

$$\begin{aligned}\Omega'^{\frac{1}{2}}\,\Omega^{-\frac{1}{2}}\,(\Omega-\Upsilon)(\Omega+\Upsilon)^{-1}\,\Omega\,(\Omega-\Upsilon)^{-1}\,(\Omega+\Upsilon)\,\Omega^{-\frac{1}{2}}\,\Omega'^{\frac{1}{2}}&\\ = \Omega'^{\frac{1}{2}}\Omega^{-\frac{1}{2}}\,.\,\Omega\,.\,\Omega^{-\frac{1}{2}}\,\Omega'^{\frac{1}{2}} = \Omega'.&\end{aligned}$$

It may be obtained by writing the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega'$$

as
$$\breve{\phi}^{-1}\,\Omega'\,\phi^{-1} = \Omega,$$

and proceeding as in the derivation of the first form; or, it may be derived from that by substituting in it for Υ the skew-symmetric matrix $\Omega'^{\frac12}\Omega^{-\frac12}\Upsilon\Omega^{-\frac12}\Omega'^{\frac12}$, as follows:—

$$\begin{aligned}
&\Omega^{-\frac12}\Omega'^{\frac12}(\Omega'-\Omega'^{\frac12}\Omega^{-\frac12}\Upsilon\Omega^{-\frac12}\Omega'^{\frac12})^{-1}(\Omega'+\Omega'^{\frac12}\Omega^{-\frac12}\Upsilon\Omega^{-\frac12}\Omega'^{\frac12})\\
&\qquad=\Omega^{-\frac12}\Omega'^{\frac12}.\Omega'^{-\frac12}(1-\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})^{-1}\Omega'^{-\frac12}.\Omega'^{\frac12}(1+\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})\Omega^{\frac12}\\
&\qquad=\Omega^{-\frac12}(1-\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})^{-1}\Omega^{-\frac12}.\Omega^{\frac12}(1+\Omega^{-\frac12}\Upsilon\Omega^{-\frac12})\Omega^{\frac12}.\Omega^{-\frac12}\Omega'^{\frac12}\\
&\qquad=(\Omega-\Upsilon)^{-1}(\Omega+\Upsilon)\,\Omega^{-\frac12}\Omega'^{\frac12}.
\end{aligned}$$

The expression $\qquad \pm\Omega^{-\frac12}(1-\Upsilon)^{-1}(1+\Upsilon)\,\Omega'^{\frac12}$

is another form of the general solution; it may be derived from the first form by substituting in that for Υ the skew-symmetric matrix $\Omega'^{\frac12}\Upsilon\Omega'^{\frac12}$.

4. *Determination of the real linear transformations of one quadric into another.*

The expressions for ϕ given in the last section will, in general, be imaginary. If, however, Ω and Ω' are real and have the same number of positive latent roots,* there are real values of ϕ satisfying the equation

$$\breve{\phi}\,\Omega\,\phi=\Omega'.$$

These can be obtained through the representation of Ω and Ω' in what may be termed their canonical or standard forms.

Let Ω be a real symmetric matrix, and let $g_1, g_2, \dots g_m$ denote its positive latent roots, and $-g_{m+1}, -g_{m+2}, \dots g_n$ its negative latent roots;* moreover, let

$$\{c_1, c_2, \dots c_n\}$$

denote a matrix whose constituents are all zero, except those in the principal diagonal, which are severally equal to c_1, c_2, &c. Then, by a well-known theorem, a real orthogonal matrix ϖ_1 can always be found such that

$$\Omega=\breve{\varpi}_1\,\theta\,\varpi_1,$$

where $\qquad \theta=\{g_1, g_2, \dots g_m, -g_{m+1}, \dots -g_n\}.$

The right-hand member of the preceding equation I term the canonical form of Ω. Similarly, if Ω' is a real symmetric matrix

* It is assumed that $|\,\Omega\,|$, the determinant of the quadric, is not zero; this is equivalent to the assumption that none of the latent roots of Ω are zero. Since Ω is real and symmetric, its latent roots are all real.

whose positive latent roots are $h_1, h_2, \dots h_m$, and whose negative latent roots are $-h_{m+1}, -h_{m+2}, \dots -h_n$, a real orthogonal matrix ϖ_2 can always be found such that

$$\Omega' = \varpi_2 \eta \breve{\varpi}_2,$$

where
$$\eta = \{h_1, h_2, \dots h_m, -h_{m+1}, \dots -h_n\}.$$

Substituting these expressions for Ω and Ω' in the equation

$$\breve{\phi} \Omega \phi = \Omega',$$

we have
$$\breve{\phi} \breve{\varpi}_1 \theta \varpi_1 \phi = \varpi_2 \eta \breve{\varpi}_2,$$

i.e.,
$$\breve{\varpi}_2 \breve{\phi} \breve{\varpi}_1 \theta \varpi_1 \phi \varpi_2 = \eta.$$

Denoting $\varpi_1 \phi \varpi_2$ by ψ, this becomes

$$\breve{\psi} \theta \psi = \eta,$$

of which the general solution, by the preceding section, is

$$\psi = \pm \theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} (\eta - \Upsilon)^{-1} (\eta + \Upsilon),$$

where Υ is an arbitrary skew-symmetric matrix, and we may take

$$\theta^{-\frac{1}{2}} = \left\{ \frac{1}{g_1^{\frac{1}{2}}}, \frac{1}{g_2^{\frac{1}{2}}}, \dots \frac{1}{g_m^{\frac{1}{2}}}, \frac{1}{g_{m+1}^{\frac{1}{2}}\sqrt{-1}}, \dots \frac{1}{g_n^{\frac{1}{2}}\sqrt{-1}} \right\},$$

$$\eta^{\frac{1}{2}} = \left\{ h_1^{\frac{1}{2}}, h_2^{\frac{1}{2}}, \dots h_m^{\frac{1}{2}}, h_{m+1}^{\frac{1}{2}}\sqrt{-1}, \dots h_n^{\frac{1}{2}}\sqrt{-1} \right\};$$

consequently,

$$\theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} = \left\{ \sqrt{\frac{h_1}{g_1}}, \sqrt{\frac{h_2}{g_2}}, \dots \sqrt{\frac{h_m}{g_m}}, \sqrt{\frac{h_{m+1}}{g_{m+1}}}, \dots \sqrt{\frac{h_n}{g_n}} \right\}$$

is real.

We have

$$\begin{aligned} \psi &= \pm \theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} \breve{\varpi}_2 . \varpi_2 (\eta - \Upsilon)^{-1} \breve{\varpi}_2 . \varpi_2 (\eta + \Upsilon) \breve{\varpi}_2 . \varpi_2 \\ &= \pm \theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} \breve{\varpi}_2 (\varpi_2 \eta \breve{\varpi}_2 - \varpi_2 \Upsilon \breve{\varpi}_2)^{-1} (\varpi_2 \eta \breve{\varpi}_2 + \varpi_2 \Upsilon \breve{\varpi}_2) \varpi_2 \\ &= \pm \theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} \breve{\varpi}_2 (\Omega' - \Upsilon_1)^{-1} (\Omega' + \Upsilon_1) \varpi_2, \end{aligned}$$

where
$$\Upsilon_1 = \varpi_2 \Upsilon \breve{\varpi}_2$$

is an arbitrary skew-symmetric matrix. Therefore, *the general expression for the real linear substitutions that will transform a given real*

quadric, whose matrix is Ω, *into another given real quadric, whose matrix is* Ω' (Ω *and* Ω' *having the same number of positive latent roots*), *is*

$$\breve{\varpi}_1 \theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} \breve{\varpi}_2 (\Omega' - \Upsilon)^{-1} (\Omega' + \Upsilon),$$

where $\theta^{-\frac{1}{2}}\eta^{\frac{1}{2}}$, a real matrix, and $\breve{\varpi}_1$ and ϖ_2, also real, have the meanings assigned to them above, and Υ is an arbitrary real skew-symmetric matrix.

Another form in which this expression may appear is

$$\begin{aligned}
&\breve{\varpi}_1 (\theta - \Upsilon)^{-1} (\theta + \Upsilon)\, \theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} \breve{\varpi}_2 \\
&= \breve{\varpi}_1 (\theta - \Upsilon)^{-1} \varpi_1 . \breve{\varpi}_1 (\theta + \Upsilon)\, \varpi_1 . \breve{\varpi}_1 \theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} \breve{\varpi}_2 \\
&= (\breve{\varpi}_1 \theta \varpi_1 - \breve{\varpi}_1 \Upsilon \varpi_1)^{-1} (\breve{\varpi}_1 \theta \varpi_1 + \breve{\varpi}_1 \Upsilon \varpi_1)\, \breve{\varpi}_1 \theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} \breve{\varpi}_2 \\
&= (\Omega - \Upsilon_1)^{-1} (\Omega + \Upsilon_1)\, \breve{\varpi}_1 \theta^{-\frac{1}{2}} \eta^{\frac{1}{2}} \breve{\varpi}_2,
\end{aligned}$$

where

$$\Upsilon_1 = \breve{\varpi}_1 \Upsilon \varpi_1$$

is an arbitrary real skew-symmetric matrix. The method of obtaining this form from the first is evident from the last section.

5. *Reduction of a quadric to a sum of squares.*

In this case Ω' is of the form $\{c_1, c_2, \dots c_n\}$. Thus, if it is required to transform by a real linear substitution the real quadric $\sum_1^n a_{rs} x_r x_s$, whose matrix is Ω, into the sum of squares $\sum_1^n \pm G_r \xi_r^2$, then, by the principle of inertia of quadratic forms, just so many of the G's must be positive as there are positive latent roots of Ω. As before, let $g_1, g_2, \dots g_m$ be the positive latent roots of Ω, and $-g_{m+1}, -g_{m+2}, \dots -g_n$ the negative latent roots of Ω. Since it is immaterial what subscripts appertain to the ξ's and their coefficients, it may be assumed that the first m of these coefficients are positive, the remainder being negative. If

$$\Omega = \breve{\varpi} \{g_1, g_2, \dots g_m, -g_{m+1} \dots -g_n\}\, \varpi,$$

ϖ being a real orthogonal matrix, the equation determining the transformation ϕ is

$$\begin{aligned}
&\breve{\phi} \breve{\varpi} \{g_1, g_2, \dots g_m, -g_{m+1}, \dots -g_n\}\, \varpi \phi \\
&\qquad = \{G_1, G_2, \dots G_m, -G_{m+1}, \dots -G_n\}.
\end{aligned}$$

Therefore, *the general expression for the real linear substitution that*

will transform a given real quadric into a sum of squares, the coefficients being real, is

$$\pm\breve{\varpi}\left\{\sqrt{\frac{G_1}{g_1}}, \sqrt{\frac{G_2}{g_2}}, \dots \sqrt{\frac{G_n}{g_n}}\right\}\left(\{G_1, G_2, \dots G_m, -G_{m+1}, \dots -G_n\} - \Upsilon\right)^{-1}$$

$$\times\left(\{G_1, G_2, \dots G_m, -G_{m+1}, \dots -G_n\} + \Upsilon\right),$$

where Υ is an arbitrary real skew-symmetric matrix.

Another form of the general expression is

$$\pm(\Omega-\Upsilon)^{-1}(\Omega+\Upsilon)\,\breve{\varpi}\left\{\sqrt{\frac{G_1}{g_1}}, \sqrt{\frac{G_2}{g_2}}, \dots \sqrt{\frac{G_n}{g_n}}\right\},$$

where Υ is an arbitrary real skew-symmetric matrix.

6. *Dependence upon the equation* $\breve{\psi}\psi = 1$ *of the equations* $\breve{\phi}\Omega\phi = \Omega$ *and* $\breve{\phi}\Omega\phi = \Omega'$.

The equation $$\breve{\phi}\,\Omega\,\phi = \Omega$$

may be written $$\Omega^{-\frac{1}{2}}\breve{\phi}\,\Omega^{\frac{1}{2}}\,.\,\Omega^{\frac{1}{2}}\phi\,\Omega^{-\frac{1}{2}} = 1,$$

since, if ϕ satisfies this equation, it satisfies the former, and conversely. This is also true if the two square roots $\Omega^{-\frac{1}{2}}$ and $\Omega^{\frac{1}{2}}$ are taken to be symmetric; and then, if

$$\psi = \Omega^{\frac{1}{2}}\phi\,\Omega^{-\frac{1}{2}},$$

the equation becomes $$\breve{\psi}\psi = 1.$$

Therefore $$\phi = \Omega^{-\frac{1}{2}}\psi\,\Omega^{\frac{1}{2}}$$

(where $\Omega^{-\frac{1}{2}}$ and $\Omega^{\frac{1}{2}}$ denote symmetric square roots, and ψ is an arbitrary orthogonal matrix) is the most general solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega.$$

Consequently, the problem of the automorphic linear transformation of a quadric resolves itself into that of the representation of an orthogonal matrix.

In this expression for ϕ, no generality is lost by regarding the square root of Ω^{-1}, which appears in this expression, as the reciprocal of that square root of Ω which also enters into this expression for ϕ;

for every solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega$$

is a solution of the equation

$$\Omega^{-\frac{1}{2}}\breve{\phi}\,\Omega^{\frac{1}{2}} \,.\, \Omega^{\frac{1}{2}}\phi\,\Omega^{-\frac{1}{2}} = 1,$$

in which the square roots of Ω that enter are all identical.* Similarly, without loss of generality, any square root of Ω may be taken, provided it is symmetric; and then, by a suitable choice of ψ, all other solutions of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega$$

may be obtained.† It will be assumed in what follows that the two square roots of Ω entering into the expression for ϕ are identical, in which case ϕ and ψ have the same latent roots; for then

$$\phi - g = \Omega^{-\frac{1}{2}}\psi\,\Omega^{\frac{1}{2}} - g = \Omega^{-\frac{1}{2}}(\psi - g)\,\Omega^{\frac{1}{2}},$$

and therefore $$|\,\phi - g\,| = |\,\psi - g\,|$$

for all values of g.

The equation $$\breve{\phi}\,\Omega\,\phi = \Omega',$$

may, similarly, be written

$$\Omega'^{-\frac{1}{2}}\breve{\phi}\,\Omega^{\frac{1}{2}} \,.\, \Omega^{\frac{1}{2}}\phi\,\Omega'^{-\frac{1}{2}} = 1,$$

since, if ϕ satisfies this equation, it satisfies the former, and con-

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* Moreover, if $\Omega_1^{\frac{1}{2}}$, $\Omega_2^{\frac{1}{2}}$ denote distinct symmetric square roots of Ω, and if

$$\phi = \Omega_1^{-\frac{1}{2}}\psi\,\Omega_2^{\frac{1}{2}},$$

where ψ is orthogonal, replacing ψ by $\psi\,\Omega_1^{\frac{1}{2}}\Omega_2^{-\frac{1}{2}}$, which is also orthogonal, since $\Omega_1^{\frac{1}{2}}\Omega_2^{-\frac{1}{2}}$ is orthogonal (for the transverse of $\Omega_1^{\frac{1}{2}}\Omega_2^{-\frac{1}{2}}$ is $\Omega_2^{-\frac{1}{2}}\Omega_1^{\frac{1}{2}}$, and

$$\Omega_2^{-\frac{1}{2}}\Omega_1^{\frac{1}{2}} \,.\, \Omega_1^{\frac{1}{2}}\Omega_2^{-\frac{1}{2}} = \Omega_2^{-\frac{1}{2}} \,.\, \Omega \,.\, \Omega_2^{-\frac{1}{2}} = \Omega_2^{-\frac{1}{2}}\Omega_2^{\frac{1}{2}} \,.\, \Omega_2^{\frac{1}{2}}\Omega_2^{-\frac{1}{2}} = 1),$$

and the product of two orthogonal matrices is orthogonal, the expression for ϕ becomes $\Omega_1^{-\frac{1}{2}}\psi\,\Omega_1^{\frac{1}{2}}$, in which the square roots that enter are reciprocals.

† Thus, if $\Omega_1^{\frac{1}{2}}$ and $\Omega_2^{\frac{1}{2}}$ denote two distinct symmetric square roots of Ω, and if

$$\phi = \Omega_1^{-\frac{1}{2}}\psi\,\Omega_1^{\frac{1}{2}},$$

where ψ is orthogonal, replacing ψ by $\Omega_1^{\frac{1}{2}}\Omega_2^{-\frac{1}{2}}\psi\,\Omega_2^{\frac{1}{2}}\Omega_1^{-\frac{1}{2}}$, which is also orthogonal, since $\Omega_1^{\frac{1}{2}}\Omega_2^{-\frac{1}{2}}$ and $\Omega_2^{\frac{1}{2}}\Omega_1^{-\frac{1}{2}}$ are orthogonal, ϕ becomes

$$\Omega_2^{-\frac{1}{2}}\psi\,\Omega_2^{\frac{1}{2}}.$$

versely. The matrices $\Omega'^{-\frac{1}{2}}$ and $\Omega^{\frac{1}{2}}$ may be taken to be symmetric; and then, if

$$\psi = \Omega^{\frac{1}{2}}\phi\,\Omega'^{-\frac{1}{2}},$$

the equation becomes $\breve{\psi}\psi = 1.$

Therefore $\phi = \Omega^{-\frac{1}{2}}\psi\,\Omega'^{\frac{1}{2}},$

in which $\Omega^{-\frac{1}{2}}$ and $\Omega'^{\frac{1}{2}}$ are any pair of symmetric square roots of Ω^{-1} and Ω', respectively, and ψ is an arbitrary orthogonal matrix, is the most general solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega'.$$

In what follows I shall give two expressions for orthogonal matrices, and their applications to the equations considered. Of these solutions of the equation

$$\breve{\psi}\psi = 1,$$

the second is absolutely general, and, therefore, gives rise to expressions which contain every solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega,$$

and every solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega' \quad \text{(see § 7).}$$

The first representation of an orthogonal matrix which will be considered is Cayley's; it gives rise to the solutions already presented in § 2 and § 3.

Thus, if -1 is not a latent root of the orthogonal matrix ψ, we may put

$$\Upsilon = \frac{\psi-1}{\psi+1};$$

whence follows

$$\breve{\Upsilon} = \frac{1}{\breve{\psi}+1}(\breve{\psi}-1) = \frac{1}{\psi^{-1}+1}(\psi^{-1}-1) = \frac{1}{1+\psi}(1-\psi) = \frac{1-\psi}{1+\psi} = -\Upsilon,$$

and $$\psi = (1-\Upsilon)^{-1}(1+\Upsilon).*$$

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* This expression for ψ is possible, for

$$|\Upsilon-1| = \left|\frac{\psi-1}{\psi+1}-1\right| = \frac{1}{|\psi+1|}\,|\overline{\psi-1}-\overline{\psi+1}| = \frac{|2|}{|\psi+1|} \neq 0;$$

therefore, $\Upsilon-1$ has a reciprocal.

This is Cayley's well-known representation of an orthogonal matrix in terms of a skew-symmetric matrix. If, however, -1 is a latent root of ψ, Υ cannot be thus expressed in terms of ψ.

If -1 is a latent root of ψ, but not $+1$, $-\psi$ will be an orthogonal matrix of which $+1$ is a latent root, but not -1;* therefore $-\psi$ is representable as above, giving

$$\psi = -(1-\Upsilon)^{-1}(1+\Upsilon).$$

Therefore, the expression

$$\pm(1-\Upsilon)^{-1}(1+\Upsilon)$$

will, for a proper choice of the skew-symmetric matrix Υ, give every orthogonal matrix, except those of which both ± 1 are latent roots.

Substituting, for ψ, in the equation

$$\phi = \Omega^{-\frac{1}{2}}\psi\,\Omega^{\frac{1}{2}},$$

Cayley's expression for an orthogonal matrix, we obtain the solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega$$

given in § 2. Thus

$$\begin{aligned}\phi &= \pm\,\Omega^{-\frac{1}{2}}(1-\Upsilon)^{-1}(1+\Upsilon)\,\Omega^{\frac{1}{2}}\\ &= \pm\,\Omega^{-\frac{1}{2}}(1-\Upsilon)^{-1}\,\Omega^{-\frac{1}{2}}\,.\,\Omega^{\frac{1}{2}}(1+\Upsilon)\,\Omega^{\frac{1}{2}}\\ &= \pm\,(\Omega-\Upsilon_1)^{-1}(\Omega+\Upsilon_1),\end{aligned}$$

in which

$$\Upsilon_1 = \Omega^{\frac{1}{2}}\Upsilon\,\Omega^{\frac{1}{2}}$$

is an arbitrary skew-symmetric matrix. And, since the latent roots of ϕ and of ψ are identical, this expression for ϕ (as stated in § 2) contains every solution of this equation, except those which have as latent roots both ± 1.

Similarly, substituting $(1-\Upsilon)^{-1}(1+\Upsilon)$ for ψ in $\Omega^{-\frac{1}{2}}\psi\,\Omega'^{\frac{1}{2}}$, we obtain the solution given in § 3, viz.,

$$\begin{aligned}\phi &= \pm\,\Omega^{-\frac{1}{2}}(1-\Upsilon)^{-1}(1+\Upsilon)\,\Omega'^{\frac{1}{2}}\\ &= \pm\,\Omega^{-\frac{1}{2}}\,\Omega'^{\frac{1}{2}}\,.\,\Omega'^{-\frac{1}{2}}(1-\Upsilon)^{-1}\,\Omega'^{-\frac{1}{2}}\,.\,\Omega'^{\frac{1}{2}}(1+\Upsilon)\,\Omega'^{\frac{1}{2}}\\ &= \pm\,\Omega^{-\frac{1}{2}}\,\Omega'^{\frac{1}{2}}(\Omega'-\Upsilon_1)^{-1}(\Omega'+\Upsilon_1),\end{aligned}$$

in which

$$\Upsilon_1 = \Omega'^{\frac{1}{2}}\Upsilon\,\Omega'^{\frac{1}{2}}$$

* See p. 296.

is an arbitrary skew-symmetric matrix. This expression contains every solution of the equation

$$\breve{\phi}\,\Omega\,\phi = \Omega\,,$$

except those for which $\Omega'^{-\frac{1}{2}}\Omega^{\frac{1}{2}}\phi$ has both ± 1 as latent roots.

7. *Solutions of the equations* $\breve{\phi}\,\Omega\,\phi = \Omega$ *and* $\breve{\phi}\,\Omega\,\phi = \Omega'$, *based on Tait's representation of an orthogonal matrix.*

Any matrix of non-vanishing determinant is separable into a product of a symmetric matrix and an orthogonal matrix. Thus, if χ is any matrix,

$$\chi = \left(\chi\frac{1}{\sqrt{\breve{\chi}\chi}}\right)\sqrt{\breve{\chi}\chi} = \frac{\chi}{\sqrt{\breve{\chi}\chi}}\sqrt{\breve{\chi}\chi}\,;$$

but $\breve{\chi}\chi$ is symmetric, therefore $\sqrt{\breve{\chi}\chi}$ may be taken to be symmetric; and, if

$$\psi = \frac{\chi}{\sqrt{\breve{\chi}\chi}},$$

then $$\breve{\psi}\psi = \frac{1}{\sqrt{\breve{\chi}\chi}}\breve{\chi}\cdot\chi\frac{1}{\sqrt{\breve{\chi}\chi}} = \frac{1}{\sqrt{\breve{\chi}\chi}}\sqrt{\breve{\chi}\chi}\sqrt{\breve{\chi}\chi}\frac{1}{\sqrt{\breve{\chi}\chi}} = 1.*$$

The function $\frac{\chi}{\sqrt{\breve{\chi}\chi}}$ will, for a proper choice of χ, be equal to any orthogonal matrix. For, if ψ is any orthogonal matrix, and if we put

$$\chi = \psi\omega,$$

where ω is any symmetric matrix whose determinant does not vanish, one value of this function of χ is ψ; thus,

$$\frac{\chi}{\sqrt{\breve{\chi}\chi}} = \frac{\psi\omega}{\sqrt{\omega\breve{\psi}\,.\,\psi\omega}} = \frac{\psi\omega}{\sqrt{\omega^2}} = \psi.$$

Therefore, substituting for ψ in $\Omega^{-\frac{1}{2}}\psi\Omega^{\frac{1}{2}}$, *the most general solution of the matrical equation* $\breve{\phi}\Omega\phi = \Omega$ *is*

$$\phi = \Omega^{-\frac{1}{2}}\frac{\chi}{\sqrt{\breve{\chi}\chi}}\Omega^{\frac{1}{2}},$$

* Kelland and Tait's *Quaternions*, Chap. x.

in which $\Omega^{-\frac{1}{2}}$ and $\Omega^{\frac{1}{2}}$ are symmetric square roots, and χ is an arbitrary matrix.

Similarly, *the most general solution of the matrical equation* $\breve{\phi}\,\Omega\,\phi = \Omega'$ *is*

$$\phi = \Omega^{-\frac{1}{2}} \frac{\chi}{\sqrt{\breve{\chi}\chi}} \Omega'^{\frac{1}{2}},$$

in which $\Omega^{-\frac{1}{2}}$ and $\Omega'^{\frac{1}{2}}$ are any pair of symmetric square roots of Ω^{-1} and Ω', respectively, and χ is arbitrary.

Expressions for $\dfrac{\chi}{\sqrt{\breve{\chi}\chi}}$ may be obtained by means of Sylvester's formula.

On a Theorem for Confocal Bicircular Quartics and Cyclides, corresponding to Ivory's Theorem for Confocal Conics and Conicoids. *By* A. L. Dixon, M.A., Fellow of Merton College, Oxford. Received and read May 11th, 1893.

Darboux (*Mémoires de l'Acad. des Sciences de Bordeaux*, t. VIII. and IX.) and Larmor (*Proc. Lond. Math. Soc.*, XVI., p. 198) have discussed Ivory's theorem and its extension, in which $\dfrac{PQ^2}{\phi(P)\,\phi(Q)}$, where $\phi(P)$ is a function of the position of P, is unaltered when for P and Q are substituted the corresponding pair of points, and have shown that, if $\phi(P)$ is a constant, P and Q lie on confocal conicoids, and that, if such a relation hold at all, P and Q lie on confocal cyclides (*i.e.*, quartic surfaces having the imaginary circle at infinity as a double line), or, if all the points are in one plane, on confocal bicircular quartics.

I propose in this paper to find $\phi(P)$, for bicircular quartics and for cyclides. The system of coordinates used has been already investigated by Darboux and Casey.

I have added a proof of the theorem for the particular case

system of confocal Cartesian ovals, derived from Greenhill's equation to such a system, viz.,

$$x+y\iota = \mathrm{sn}^2\,\tfrac{1}{2}\,(u+v\iota).$$

PART I.

1. Let $X=0$, $Y=0$, $Z=0$ be the equations to three mutually orthogonal circles, where

$$X \equiv x^2+y^2-2l_1x-2m_1y+d_1, \text{ \&c., \&c.,}$$

with the three conditions

$$d_2+d_3 = 2l_2l_3+2m_2m_3, \text{ \&c., \&c.}$$

Then the equation of the circle (V) which is orthogonal to all three is given by

$$\begin{vmatrix} r^2 & x & y & 1 \\ d_1 & l_1 & m_1 & 1 \\ d_2 & l_2 & m_2 & 1 \\ d_3 & l_3 & m_3 & 1 \end{vmatrix} = 0 \quad [\text{where } r^2 \equiv x^2+y^2] \;\ldots(\mathrm{A}).$$

If we multiply this determinant by

$$\begin{vmatrix} 1 & -2x & -2y & r^2 \\ 1 & -2l_1 & -2m_1 & d_1 \\ 1 & -2l_2 & -2m_2 & d_2 \\ 1 & -2l_3 & -2m_3 & d_3 \end{vmatrix} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{B}),$$

which is only another form of V, we get, as is well known, the square of V in the form

$$\begin{vmatrix} 0 & X & Y & Z \\ X & -2r_1^2 & 0 & 0 \\ Y & 0 & -2r_2^2 & 0 \\ Z & 0 & 0 & -2r_3^2 \end{vmatrix}$$

from which we get the identity

$$\frac{X^2}{r_1^2}+\frac{Y^2}{r_2^2}+\frac{Z^2}{r_3^2}+\frac{V^2}{r_4^2} = 0.$$

But, if we write $x'y'$ for xy in (B), and then multiply, we shall get, putting ρ^2 for $\overline{x-x'}^2+\overline{y-y'}^2$,

$$\begin{vmatrix} \rho^2 & X & Y & Z \\ X' & -2r_1^2 & 0 & 0 \\ Y' & 0 & -2r_2^2 & 0 \\ Z' & 0 & 0 & -2r_3^2 \end{vmatrix},$$

from which we get the expression for the distance between two points

$$-2\rho^2 = \frac{XX'}{r_1^2}+\frac{YY'}{r_2^2}+\frac{ZZ'}{r_3^2}+\frac{VV'}{r_4^2}.$$

It will be convenient to alter the notation, and write X instead of $\frac{X}{r_1}$, &c., &c., so that the identical relation is now written

$$X^2+Y^2+Z^2+V^2 = 0 \text{.........................(C)},$$

and the formula for the distance between two points

$$-2\rho^2 = XX'+YY'+ZZ'+VV' \text{ ....(D)}.$$

2. The equation to any bicircular quartic can be written in the form

$$aX^2+bY^2+cZ^2 = 0,\text{*}$$

where X, Y, Z are defined as above, the number of independent constants being eight, and the terms of the third and fourth degrees of the right form, and further, from the identical relation (C), we see that this can be done in four ways.

Let us now consider what relations hold between the parameters in order that

$$\frac{X^2}{A}+\frac{Y^2}{B} = Z^2 \text{ ....(1)}$$

may represent a system of confocal curves.

The curve is the envelope of

$$lX+mY = Z \text{ ....(2)},$$

where $$Al^2+Bm^2 = 1 \text{ ....(3)}.$$

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* Casey, *Transactions of the Royal Irish Academy*, Vol. xxiv., 1869.

The condition that (2) should reduce to a point circle, and therefore represent a focus, is

$$\left(l\frac{d_1}{r_1}+m\frac{d_2}{r_2}-\frac{d_3}{r_3}\right)\left(\frac{l}{r_1}+\frac{m}{r_2}-\frac{1}{r_3}\right)$$
$$=\left(l\frac{l_1}{r_1}+m\frac{l_2}{r_2}-\frac{l_3}{r_3}\right)^2\left(l\frac{m_1}{r_1}+m\frac{m_2}{r_2}-\frac{m_3}{r_3}\right)^2,$$

which reduces at once to

$$l^2+m^2+1=0 \quad\text{.............................(4)},$$

since X, Y, Z are mutually orthogonal.

Therefore, being given one curve of the system, we may determine l^2 and m^2 from (3) and (4), and, in order that any other curve

$$\frac{X^2}{A'}+\frac{Y^2}{B'}=Z^2$$

should have the same foci, it is necessary and sufficient that

$$A'l^2+B'm^2=1,$$

where l^2 and m^2 are constants independent of A' and B', or otherwise that

$$B'=A'p+q,$$

where p and q are constant, or again that

$$\frac{B'-B}{A'-A}=\text{const.}$$

3. It is easily seen from this that, as is well known, these confocals cut at right angles, for, if

$$l_1X+m_1Y=Z,$$
$$l_2X+m_2Y=Z$$

be the circles touching

$$\frac{X^2}{A_1}+\frac{Y^2}{B_1}=Z^2,$$
$$\frac{X^2}{A_2}+\frac{Y^2}{B_2}=Z^2,$$

respectively, at their common point (x', y'),

$$l_1=\frac{X'}{A_1Z'},\quad m_1=\frac{Y'}{B_1Z'},\qquad l_2=\frac{X'}{A_2Z'},\quad m_2=\frac{Y'}{B_2Z'},$$
$$A_1l_1^2+B_1m_1^2=1,\quad A_2l_2^2+B_2m_2^2=1;$$

from which we derive

$$A_1 l_1 l_2 + B_1 m_1 m_2 = 1,$$

$$A_2 l_1 l_2 + B_2 m_1 m_2 = 1;$$

and therefore, by the help of any of the relations of the last article,

$$l_1 l_2 + m_1 m_2 + 1 = 0,$$

which is seen to be the condition that the two tangent circles cut orthogonally, being deduced in exactly the same way as (4).

4. Also the four foci given by (3) and (4) all lie on the circle V, as may be shown by expressing the condition (4) in terms of the coordinates of the point circle given by (2), when it reduces to $V = 0$, or otherwise, since V cuts orthogonally any circle of the form (2).

5. Returning to the equation

$$\frac{X^2}{A} + \frac{Y^2}{B} = Z^2,$$

with the condition $\qquad B = Ap + q,$

we get, to determine the parameters of the two curves of the series through any given point,

$$X'^2 (Ap+q) + Y'^2 A - Z'^2 A (Ap+q) = 0,$$

and, if $A'A''$ be the roots of this,

$$\frac{qX'^2}{pZ'^2} = A'A''.$$

Therefore, defining corresponding points on two curves (1) and (2) as the intersections with them of a third curve of the system which cuts them both orthogonally, we get, for corresponding points P and Q,

$$\frac{X_P}{Z_P \sqrt{A_1}} = \frac{X_Q}{Z_Q \sqrt{A_2}}.$$

Similarly, for another pair (P', Q') of corresponding points, we shall get

$$\frac{X_{P'}}{Z_{P'} \sqrt{A_1}} = \frac{X_{Q'}}{Z_{Q'} \sqrt{A_2}};$$

and therefore $\qquad \dfrac{X_P X_{Q'}}{Z_P Z_{Q'}} = \dfrac{X_Q X_{P'}}{Z_Q Z_{P'}},$

and of course the same relations in Y and V.

6. Therefore, using the formula already found,

$$-2\rho^2 = XX' + YY' + ZZ' + VV',$$

we see that the ratios

$$PQ'^2 : X_P X_{Q'} : Y_P Y_{Q'} : Z_P Z_{Q'} : V_P V_{Q'}$$

are equal to the ratios

$$P'Q^2 : X_{P'} X_Q : Y_{P'} Y_Q : Z_{P'} Z_Q : V_{P'} V_Q,$$

or that the ratio of the distance of any two points to the product of lengths of the tangents drawn from them to any one of the four focal circles of the system is the same as for the pair of corresponding points.

7. Exactly in the same way as for circles, we shall get, if X, Y, Z, V, W represent five mutually orthogonal spheres, an identity which we write as

$$X^2 + Y^2 + Z^2 + V^2 + W^2 = 0,$$

and further a formula for the distance between two points,

$$\rho^2 = -\tfrac{1}{2}\left\{XX' + YY' + ZZ' + VV' + WW'\right\}.$$

The equation of any cyclide, *i.e.*, a surface with the imaginary circle at infinity as a double line, can be written

$$\frac{X^2}{A} + \frac{Y^2}{B} + \frac{Z^2}{C} = V^2 \quad \text{.....................(1),*}$$

since there are thirteen independent constants.

This surface is the envelope of

$$lX + mY + nZ = V \quad \text{........................(2),}$$

with the condition $\quad Al^2 + Bm^2 + Cn^2 = 1$ ....(3),

and, when the equation reduces to a point circle and represents a focus, we have

$$l^2 + m^2 + n^2 + 1 = 0 \quad \text{........................(4),}$$

which gives the focal curve as the intersection of two surfaces (3) and (4) [the surface represented by (4) being the sphere W].

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* *Cf.* Darboux, "Sur une Classe remarquable de Courbes et de Surfaces Algébriques," *Mémoires de l'Acad. des Sciences de Bordeaux*, t. IX., and Casey, "Cyclides and Spheroquartics," *Phil. Trans.*, Vol. CLXI., 1871.

8. If $$\frac{X^2}{A'}+\frac{Y^2}{B'}+\frac{Z^2}{C'}=V^2$$

is a confocal, $$l^2+m^2+n^2+1=0,$$

$$Al^2+Bm^2+Cn^2=1,$$

$$A'l^2+B'm^2+C'n^2=1$$

must have a common curve of intersection; and therefore the four determinants

$$\begin{Vmatrix} A' & A & 1 \\ B' & B & 1 \\ C' & C & 1 \\ 1 & 1 & -1 \end{Vmatrix}$$

must vanish; and therefore, if we choose p and q so that

$$1=p-q,$$

$$A=Bp+q,$$

we must have also $$A'=B'p+q,$$

the same relation as for the parameters of confocal curves, and we shall get, just as for them, that

$$\frac{X'}{V'\sqrt{A'}}=\frac{X''}{V''\sqrt{A''}},\ \text{\&c., \&c.}$$

for corresponding points, and finally that the ratio of the distance between any two points to the product of the lengths of the tangents drawn from them to any one of the five focal spheres is the same as for the pair of corresponding points.

9. It is worth noticing* that it follows at once that if two surfaces have one spherical focal curve common they have all five. For, if the same surface be written

$$\frac{X^2}{A}+\frac{Y^2}{B}+\frac{Z^2}{C}=V^2,$$

and $$\frac{X^2}{a}+\frac{Y^2}{b}+\frac{Z^2}{c}=W^2,$$

[* Casey seems to have been in error on this point. *Cf. loc. cit.*, § 117.]

the identity (C) gives us

$$\frac{1}{A}+\frac{1}{a}=-1,\quad \frac{1}{B}+\frac{1}{b}=-1,\ \&c.;$$

and therefore, if $$A = Bp+q,$$

where $$1 = p-q,$$

we have $$\frac{a}{a+1}=\frac{b}{b+1}p-q,$$

$$ab+a = (ab+b)\,p-q\,(ab+a+b+1),$$

or $$a\,(1-q) = b\,(p-q)-q,$$

with the relation $$1-q = p-q-q,$$

which establishes the proposition.

10. In the particular case of a symmetrical cyclide or bicircular quartic, in which one of the focal spheres or circles becomes a plane or line, the power of the sphere or circle becomes twice the perpendicular distance from the plane or line, and for such surfaces and curves we have that

$$\frac{PQ'^2}{y_P\,y_{Q'}}=\frac{P'Q^2}{y_{P'}\,y_Q},$$

where y is the perpendicular distance from such plane or line.

In the case of binodal cyclides, &c., the focal spheres reduce to points, the formula being obtained at once by inversion from Ivory's heorem.

Part II. *Connexion of the Theorem with Elliptic Integrals.*

1. If $$x+y\iota = \mathrm{sn}^2\tfrac{1}{2}\,(u+\iota v)$$

(Greenhill, *Ell. Functions*, Chap. VIII.),

$$u = \text{const.}\quad \text{and}\quad v = \text{const.}$$

give a system of mutually orthogonal Cartesian ovals.

Take the two pairs of corresponding points given by (u_1, v_1), (u_2, v_2) and (u_1, v_2), (u_2, v_1).

Let r be the distance between the two points P_1, P_2 given by

$$x_1 \pm y_1\iota = \mathrm{sn}^2\tfrac{1}{2}\,(u_1 \pm \iota v_1),$$

$$x_2 \pm y_2\iota = \mathrm{sn}^2\tfrac{1}{2}\,(u_2 \pm \iota v_2).$$

Let r_1, r_2 be the distances of P_1 and P_2 from a focus (say the sn-focus O').

Let $$s_1 \equiv \operatorname{sn}\tfrac{1}{2}(u_1+\iota v_1), \quad s_1' \equiv \operatorname{sn}\tfrac{1}{2}(u_1-\iota v_1),$$
$$s_2 \equiv \operatorname{sn}\tfrac{1}{2}(u_2+\iota v_2), \quad s_2' \equiv \operatorname{sn}\tfrac{1}{2}(u_2-\iota v_2).$$

Then $$P_1P_2^2 = r^2 = \overline{x_1-x_2}^2+\overline{y_1-y_2}^2$$
$$= \left\{\operatorname{sn}^2\tfrac{1}{2}(u_1+\iota v_1)-\operatorname{sn}^2\tfrac{1}{2}(u_2+\iota v_2)\right\}\left\{\operatorname{sn}^2\tfrac{1}{2}(u_1-\iota v_1)-\operatorname{sn}^2\tfrac{1}{2}(u_2-\iota v_2)\right\}$$
$$= \operatorname{sn}\tfrac{1}{2}\left\{u_1+u_2+\iota(v_1+v_2)\right\}\operatorname{sn}\tfrac{1}{2}\left\{u_1-u_2+\iota(v_1-v_2)\right\}$$
$$\operatorname{sn}\tfrac{1}{2}\left\{u_1-u_2-\iota(v_1-v_2)\right\}\operatorname{sn}\tfrac{1}{2}\left\{u_1+u_2-\iota(v_1+v_2)\right\}$$
$$\times\left\{1-\kappa^2 s_1^2 s_2^2\right\}\times\left\{1-\kappa^2 s_1'^2 s_2'^2\right\}$$
$$= ADD', \text{ say,}$$

where A is unaltered by the interchange of u_1 and u_2, *i.e.*, by transference to corresponding points.

Also $$D = 1-\kappa^2 s_1^2 s_2^2$$
$$= 1-\kappa^2(x_1+y_1\iota)(x_2+y_2\iota)$$
$$= 1-\kappa^2 r_1 r_2\left\{\cos\overline{\theta_1+\theta_2}+\iota\sin\overline{\theta_1+\theta_2}\right\},$$
$$D' = 1-\kappa^2 r_1 r_2\left\{\cos\overline{\theta_1+\theta_2}-\iota\sin\overline{\theta_1+\theta_2}\right\};$$

therefore $$DD' = 1-2\kappa^2 r_1 r_2\cos\overline{\theta_1+\theta_2}+\kappa^4 r_1^2 r_2^2$$
$$= 1-\kappa^2(r_1^2+r_2^2-r'^2)+\kappa^4 r_1^2 r_2^2$$
$$= (1-\kappa^2 r_1^2)(1-\kappa^2 r_2^2)+\kappa^2 r'^2,$$

where r' is the distance between P_1 and the image of P_2 in the focal axis, *i.e.*, between the points (u_1, v_1) and $(u_2, -v_2)$; and therefore
$$\frac{(1-\kappa^2 r_1^2)(1-\kappa^2 r_2^2)+\kappa^2 r'^2}{r^2} = \frac{1}{A},$$
and is unaltered when u_1 and u_2 are interchanged.

Further, we shall have that
$$\frac{(1-\kappa^2 r_1^2)(1-\kappa^2 r_2^2)+\kappa^2 r^2}{r'^2}$$
is unaltered, by considering that (u_1, v_1), $(u_2, -v_2)$ and $(u_1, -v_2)$, (u_2, v_1) are corresponding points.

2. Writing B for $(1-\kappa^2r_1^2)(1-\kappa^2r_2^2)$, we have then, that

$$\frac{B+\kappa^2r'^2}{r^2} \text{ and } \frac{B+\kappa^2r^2}{r'^2}$$

are unchanged; and therefore also their sum, difference, and product, *i.e.*,

$$\frac{B\,(r^2+r'^2)+\kappa^2\,(r^4+r'^4)}{r^2r'^2},$$

$$\frac{B+\kappa^2\,(r^2+r'^2)}{r^2r'^2}\,(r^2-r'^2),$$

$$\frac{B^2+B\kappa^2\,(r+r'^2)+\kappa^4r^2r'^2}{r^2r'^2},$$

or, adding $2\kappa^2$ to the sum and subtracting κ^4 from the product, that

$$(r^2+r'^2)\left\{\frac{B+\kappa^2\,(r^2+r'^2)}{r^2r'^2}\right\}\,(r^2-r'^2)\left\{\frac{B+\kappa^2\,(r^2+r'^2)}{r^2r'^2}\right\},$$

$$B\,\frac{B+\kappa^2\,(r^2+r'^2)}{r^2r'^2}$$

are unchanged; and therefore the ratios

$$\frac{r^2+r'^2}{B} \text{ and } \frac{r^2-r'^2}{B};$$

or, finally, that $\qquad \dfrac{r^2}{B}$, or $\dfrac{r^2}{(1-\kappa^2r_1^2)(1-\kappa^2r_2^2)}$,

is the same as for the pair of corresponding points.

3. Since $\qquad r^2-r'^2 = -4y_1y_2 = (s_1^2-s_1'^2)(s_2^2-s_2'^2)$,

we get at once

$$\frac{r^2-r'^2}{(1-\kappa^2r_1^2)(1-\kappa^2r_2^2)} = \frac{s_1^2-s_1'^2}{1-\kappa^2s_1^2s_1'^2}\cdot\frac{s_2^2-s_2'^2}{1-\kappa^2s_2^2s_2'^2}$$

$$= \operatorname{sn} u_1 \operatorname{sn} \iota v_1 \operatorname{sn} u_2 \operatorname{sn} \iota v_2;$$

and therefore, since $A = \dfrac{r^2}{(1-\kappa^2r_1^2)(1-\kappa^2r_2^2)+\kappa^2r'^2}$,

$$\frac{r^2}{(1-\kappa^2r_1^2)(1-\kappa^2r_2^2)} = A\,\frac{1-\kappa^2 \operatorname{sn} u_1 \operatorname{sn} \iota v_1 \operatorname{sn} u_2 \operatorname{sn} \iota v_2}{1-\kappa^2A},$$

where, as before,

$$A = \operatorname{sn}\tfrac{1}{2}\left\{u_1+u_2+\iota\,(v_1+v_2)\right\}\operatorname{sn}\tfrac{1}{2}\left\{u_1+u_2-\iota\,(v_1+v_2)\right\}$$
$$\times\operatorname{sn}\left\{(u_1-u_2)+\iota\,(v_1-v_2)\right\}\operatorname{sn}\left\{(u_1-u_2)-\iota\,(v_1-v_2)\right\}.$$

4. The direct verification of this is interesting; we have from § 1,

$$\frac{r^2}{A(1-\kappa^2 r_1^2)(1-\kappa^2 r_2^2)} = \frac{(1-\kappa^2 s_1^2 s_2^2)(1-\kappa^2 s_1'^2 s_2'^2)}{(1-\kappa^2 s_1^2 s_1'^2)(1-\kappa^2 s_2^2 s_2'^2)}.$$

Introduce the factor

$$(1-\kappa^2 s_1^2 s_1'^2)(1-\kappa^2 s_2^2 s_2'^2)-\kappa^2(s_1^2-s_1'^2)(s_2^2-s_2'^2)$$
$$\equiv (1-\kappa^2 s_1^2 s_2^2)(1-\kappa^2 s_1'^2 s_2'^2)-\kappa^2(s_1^2-s_2^2)(s_1'^2-s_2'^2),$$

and we get

$$\frac{1-\kappa^2\dfrac{s_1^2-s_1'^2}{1-\kappa^2 s_1^2 s_1'^2}\cdot\dfrac{s_2^2-s_2'^2}{1-\kappa^2 s_2^2 s_2'^2}}{1-\kappa^2\dfrac{s_1^2-s_2^2}{1-\kappa^2 s_1^2 s_2^2}\cdot\dfrac{s_1'^2-s_2'^2}{1-\kappa^2 s_1'^2 s_2'^2}},$$

or

$$\frac{1-\kappa^2 \operatorname{sn} u_1 \operatorname{sn} \iota v_1 \operatorname{sn} u_2 \operatorname{sn} \iota v_2}{\begin{array}{c}1-\kappa^2 \operatorname{sn}\frac{1}{2}\left\{(u_1+u_2)+\iota(v_1+v_2)\right\}\operatorname{sn}\frac{1}{2}\left\{(u_1+u_2)-\iota(v_1+v_2)\right\}\\ \times\operatorname{sn}\frac{1}{2}\left\{(u_1-u_2)+\iota(v_1-v_2)\right\}\operatorname{sn}\frac{1}{2}\left\{(u_1-u_2)-\iota(v_1-v_2)\right\}\end{array}}.$$

The identity here found may be written in the form

$$\frac{1-\kappa^2\operatorname{sn}^2(a+b)\operatorname{sn}^2(c+d)\,.\,1-\kappa^2\operatorname{sn}^2(a-b)\operatorname{sn}^2(c-d)}{1-\kappa^2\operatorname{sn}^2(a+b)\operatorname{sn}^2(a-b)\,.\,1-\kappa^2\operatorname{sn}^2(c+d)\operatorname{sn}^2(c-d)}$$
$$=\frac{1-\kappa^2\operatorname{sn}^2(c+b)\operatorname{sn}^2(a+d)\,.\,1-\kappa^2\operatorname{sn}^2(c-b)\operatorname{sn}^2(a-d)}{1-\kappa^2\operatorname{sn}^2(c+b)\operatorname{sn}^2(c-b)\,.\,1-\kappa^2\operatorname{sn}^2(a+d)\operatorname{sn}^2(a-d)};$$

or, again, putting the symbol

$$\overline{(1+2)(3+4)} \quad \text{for} \quad 1-\kappa^2\operatorname{sn}^2(a+b)\operatorname{sn}^2(c+d),$$

we shall get

$$\frac{\overline{(1+2)(3+4)}\,.\,\overline{(1-2)(3-4)}}{\overline{(3+2)(1+4)}\,.\,\overline{(3-2)(1-4)}} = \frac{\overline{(1+2)(1-2)}\,.\,\overline{(3+4)(3-4)}}{\overline{(3+2)(3-2)}\,.\,\overline{(3+4)(1-4)}}$$
$$= \frac{\overline{(1-2)(3+4)}\,.\,\overline{(1+2)(3-4)}}{\overline{(3-2)(1+4)}\,.\,\overline{(3+2)(1-4)}}.$$

5. So far, then, we have the result that the ratios

$$r^2 : y_1 y_2 : (1-\kappa^2 r_1^2)(1-\kappa^2 r_2^2)$$

are unaltered for pairs of corresponding points, and we may notice further that, since

$$\frac{y_1}{1-\kappa^2 r_1^2} = \operatorname{sn} u_1 \operatorname{sn} \iota v_1,$$

the ratio $\dfrac{y}{(1-\kappa^2r^2)\,\mathrm{sn}\,u}$ is the same for any two corresponding points on the ovals u_1 and u_2.

But, since
$$\mathrm{cn}\left(\kappa' u_1\ \frac{\iota\kappa}{\kappa'}\right) = \mathrm{sn}\,(K-u_1\,\kappa)$$
(Greenhill, *Ell. Func.*, p. 259, § 240), and
$$\mathrm{dn}\left(\iota\kappa u,\ \frac{\kappa'}{\iota\kappa}\right) = \mathrm{sn}\,(K-u,\,\kappa),$$
we see that these results must be also true for the other foci, changing the value of κ to $\dfrac{\iota\kappa}{\kappa'}$ and $\dfrac{1}{\kappa'}$ respectively, and in fact we have
$$\frac{\kappa'^2+\kappa^2r_1'^2}{1-\kappa^2r_1^2}\cdot\frac{\kappa'^2+\kappa^2r_2'^2}{1-\kappa^2r_2^2} = \mathrm{dn}\,u_1\,\mathrm{dn}\,\iota v_1\,\mathrm{dn}\,u_2\,\mathrm{dn}\,\iota v_2,$$
and also
$$\frac{\kappa'^2-\kappa^4r_1''^2}{1-\kappa^2r_1^2}\cdot\frac{\kappa'^2-\kappa^4r_2''^2}{1-\kappa^2r_2^2} = \kappa^4\,\mathrm{cn}\,u_1\,\mathrm{cn}\,\iota v_1\,\mathrm{cn}\,u_2\,\mathrm{cn}\,\iota v_2.$$

6. The result may also be stated thus: The ratios
$$r^2 : y_1y_2 : (a^2-r_1^2)(a^2-r_2^2)$$
are unaltered for pairs of corresponding points, where a is the radius of the degenerate circle of the system which has the chosen focus as centre, or a^2 is the product of the distances of the other two foci from the one chosen.

In this last form,
$$\frac{P_1P_2^2}{(F_1F_2\,.\,F_1F_3-F_1P_1^2)(F_1F_2\,.\,F_1F_3-F_1P_2^2)},$$
it is best adapted for inversion, and we get that for a system of bicircular quartics with a common set of concyclic foci O_1, O_2, O_3, O_4,
$$\frac{PQ^2}{(O_1O_2.O_1O_4.O_1P^2-O_1O_3.O_1O_4.O_2P^2)(O_2O_3.O_2O_4.O_1Q^2-O_1O_3.O_1O_4.O_2Q^2)}$$
is unchanged for a transition to a corresponding pair of points.

Thursday, June 8th, 1893.

A. B. BASSET, Esq., F.R.S., Vice-President, in the Chair.

The following gentlemen were elected members:—T. S. Fiske, Ph.D., Columbia College, New York, U.S.A., Secretary to the New York Mathematical Society; G. Bruce Halsted, Ph.D., Professor of Mathematics, Austin, Texas, U.S.A.; H. M. Macdonald, M.A., Fellow of Clare College, Cambridge; D. B. Mair, B.A., late Scholar of Christ's College, Cambridge.

The Chairman announced that the Council had unanimously made the fourth award of the De Morgan Medal to Professor F. Klein, of Göttingen, on the ground of his many contributions to the advance of mathematical science.

The following papers were read:—

Complex Integers derived from $\theta^3-2=0$, and on the Algebraical Integers derived from an Irreducible Cubic Equation: Prof. G. B. Mathews.

Pseudo-Elliptic Integrals, and their Dynamical Applications: Prof. Greenhill.

On the Expansion of some Infinite Products: Prof. L. J. Rogers.

Note on some Properties of Gauche Cubics: Mr. T. R. Lee.

Note on the Centres of Similitude of a Triangle of Constant Form *circumscribed* to a given Triangle: Mr. J. Griffiths.

The following presents were made to the Library:—

"Proceedings of the Royal Society," Vol. LIII., No. 321.

Blake, E. M.—"'Method of Indeterminate Coefficients and Exponents' applied to Differential Equations," 8vo; New York, 1893.

"Proceedings of the Physical Society of London," Vol. XII., Pt. 1; April, 1893.

"Jahrbuch über die Fortschritte der Mathematik," Band XXII., Heft 2, Jahrgang, 1890; Berlin, 1893.

Steiner, P.—"Pasilingua, Die Sprache von Pan-Amerika und die Universal-Sprache, auf Grund einer Neuenglischen Grammatik," Heft 1; Neuwied.

"Proceedings of the Cambridge Philosophical Society," Vol. VIII., Pt. 1; Mich. Term, 1892.

"Journal of the Institute of Actuaries," Vol. XXX., Pt. 5; April, 1893, London.

"Elektrotechnische Bibliographie," Band I., Heft 1; Leipzig, 1893.

"Bulletin de la Société Physico-Mathématique de Kasan," 2e Séries, Tome 1, No. 3; 1891.

"Wiskundige Opgaven met de Oplossingen," 7e Deel, 7de Stük; Amsterdam, 1893.

"Bulletin of the New York Mathematical Society," Vol. II., No. 8; May, 1893.

"Bulletin des Sciences Mathématiques," 2e Série, Tome XVI., Tables des Matières et Noms d'Auteurs, 1892.

"Berichte über die Verhandlungen der Konigl. Sachs. Gesellschaft der Wissenschaften zu Leipzig," I., 1898.

"Journal of the College of Science—Japan University," Vol. VI., Pt. 1; Tokyo, Japan.

"Atti della Reale Accademia dei Lincei, 5° Serie — Rendiconti," Vol. II., Fasc. 7, 1° Sem.; Roma, 1893.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 2a, Vol. VII., Fasc. 4; Napoli, 1893.

"Educational Times," June, 1893.

"Indian Engineering," Vol. XIII., Nos. 16–19.

On the Complex Integers connected with the Equation $\theta^3-2=0$. *By* G. B. Mathews.

Received May 29th, 1893. Read June 8th, 1893.

Professor Dedekind's theory of ideals, as expounded in the third edition of Dirichlet's *Vorlesungen über Zahlentheorie*, or in Darboux's *Bulletin* (1876, 1877) is of so general and abstract a character that it can hardly be appreciated as it deserves until after the application of it to particular cases. Its connexion with the theory of quadratic forms, or, which is the same thing, with the theory of algebraical integers defined by an irreducible quadratic equation, is discussed in some detail in the memoirs above referred to; in the present note I propose to illustrate the theory by applying it to the system of algebraical integers defined by the equation $\theta^3-2=0$. The reader will see that many of the results are equally true when the defining equation is $\theta^3-n=0$, where n is a positive integer different from 2; I have, however, thought it best to confine myself to the more special case, in order to make the discussion as simple and concrete as possible. References to the *Vorlesungen* will be expressed by *D.*, followed by the number of the section.

1. Let θ denote a root of the irreducible equation

$$\theta^3-2=0,$$

and, to fix the ideas, suppose $\theta=\sqrt[3]{2}$. Then, in connexion therewith, we have a *corpus* $\Omega(\theta)$ comprising all the quantities such as

$$\omega=x+y\theta+z\theta^2,$$

where x, y, z are rational numbers. It is easily proved that, if ω is an algebraical integer, x, y, z must be integral; the converse is obviously true, so that the finite modulus

$$\mathfrak{o}=\left[1, \theta, \theta^2\right]$$

(*cf.* D. 166) contains all the integers of the corpus, and only these.

The discriminant $\Delta(1, \theta, \theta^2)=2^2 . 3^3$.

There is only one fundamental unit besides ± 1, namely,

$$\epsilon=(1-\theta)$$

(*cf.* D. 177).

2. The first thing to be done is to discover a system of prime ideals. Let p be any real prime; then it follows from D. 165 that, if $\mathfrak{p}$ is an ideal factor of p, and $N(\mathfrak{p})=p$, we may write

$$\mathfrak{p}=\left[p,\ a+\theta,\ b+\theta^2\right],$$

where a and b are integers.

In order that the modulus $\left[p, a+\theta, b+\theta^2\right]$ may be an ideal, it is necessary and sufficient that the six products

$$p\theta,\quad p\theta^2,\quad (a+\theta)\,\theta,\quad (a+\theta)\,\theta^2,\quad (b+\theta^2)\,\theta,\quad (b+\theta^2)\,\theta^2$$

belong to the modulus.

Now
$$\begin{aligned} p\theta &= p\,(a+\theta)-ap,\\ p\theta^2 &= p\,(b+\theta^2)-bp,\\ (a+\theta)\,\theta &= (b+\theta^2)+a\,(a+\theta)-(b+a^2),\\ (a+\theta)\,\theta^2 &= 2+a\theta^2 = a\,(b+\theta^2)-(ab-2),\\ (b+\theta^2)\,\theta &= 2+b\theta = b\,(a+\theta)-(ab-2),\\ (b+\theta^2)\,\theta^2 &= 2\theta+b\theta^2 = b\,(b+\theta^2)+2\,(a+\theta)-(b^2+2a). \end{aligned}$$

Hence the required conditions are

$$\left.\begin{array}{l} b+a^2 \equiv 0, \\ ab-2 \equiv 0, \\ b^2+2a \equiv 0, \end{array}\right\} \pmod{p}.$$

These are equivalent to

$$\left.\begin{array}{l} a^3+2 \equiv 0, \\ b+a^2 \equiv 0, \end{array}\right\} \pmod{p},$$

which admit of solution if 2 is a cubic residue of p, but not otherwise. Hence the theorem:

If p is a real prime of which 2 is a cubic residue, then p is divisible by the ideal $\mathfrak{p} = [p,\ a+\theta,\ b+\theta^2]$, where a, b are obtained by solving the congruences $a^3+2 \equiv 0$, $b+a^2 \equiv 0$, (mod p).

When p is of the form $6n-1$, 2 is always a cubic residue of p; the primes $6n+1$, below 1000, of which 2 is a cubic residue, are, according to the *Canon Arithmeticus*,

31, 43, 109, 127, 157, 223, 229, 277, 283, 307,

397, 433, 439, 457, 499, 601, 643, 691, 727, 733,

739, 811, 919, 997;

every one of these will therefore have an ideal factor of the kind considered.

In some cases $\mathfrak{p}$ reduces to a principal ideal $\mathfrak{o}\mu$, and then p is the norm of the real number μ. For instance, $p=5$ gives

$$\begin{aligned} \mathfrak{p} &= [5,\ 2+\theta,\ 1+\theta^2] \\ &= [1+2\theta-\theta^2,\ \theta,\ 1](1+\theta^2) = [\theta^2,\ \theta,\ 1](1+\theta^2) = \mathfrak{o}(1+\theta^2), \end{aligned}$$

and hence $$5 = N(1+\theta^2).$$

On the other hand, if $p=11$,

$$\mathfrak{p} = [11,\ 4+\theta,\ 6+\theta^2],$$

which is not a principal ideal.

It may be proved in exactly the same way that—

If p is a real prime of which 2 is a cubic residue, then p is divisible by the ideal $\mathfrak{p}' = [p,\ p\theta,\ -b-a\theta+\theta^2]$, where a, b, as before, are found from the congruences $a^3+2 \equiv 0$, $b+a^2 \equiv 0$, (mod p); and $N(\mathfrak{p}') = p^2$.

Since $$N(\mathfrak{p}\mathfrak{p}') = N(\mathfrak{p})\,N(\mathfrak{p}') = p^3 = N(p),$$

it follows that $$\mathfrak{o}p = \mathfrak{p}\mathfrak{p}';$$

or, as we shall generally write it,

$$p = \mathfrak{p}\mathfrak{p}'.$$

It is easy, and rather interesting, to verify *a posteriori* that $\mathfrak{p}\mathfrak{p}'$ is actually divisible by p; that the quotient is equivalent to $\mathfrak{o}$, is evident from consideration of the norms.

For completeness' sake, it is necessary to prove that $\mathfrak{p}'$ is distinct from $\mathfrak{p}^2$. We have

$$\begin{aligned}(a+\theta)^2 &= a^2+2a\theta+\theta^2 \\ &= (-b-a\theta+\theta^2)+(a^2+b)+3a\theta,\end{aligned}$$

and this cannot be contained in $\mathfrak{p}'$ except when $3a \equiv 0$, (mod p), which only occurs when $p = 3$. It is easily found that

$$3 = N(1+\theta), \quad \mathfrak{p} = \mathfrak{o}\,(1+\theta), \quad \mathfrak{p}' = \mathfrak{o}\,(1-\theta+\theta^2) = -\epsilon\mathfrak{p}^2.$$

3. We will now consider a prime of the second class, say $p = 31$. The congruence

$$a^3+2 \equiv 0, \ (\text{mod } 31),$$

has three solutions, namely

$$a \equiv 11,\ 24,\ 27,\ (\text{mod } 31),$$

and the corresponding values of b are

$$b \equiv 3,\ 13,\ 15.$$

Hence we have the three ideals

$$\begin{aligned}\mathfrak{p}_1 &= [31,\ 11+\theta,\ \ 3+\theta^2], \\ \mathfrak{p}_2 &= [31,\ 24+\theta,\ 13+\theta^2], \\ \mathfrak{p}_3 &= [31,\ 27+\theta,\ 15+\theta^2].\end{aligned}$$

These are obviously all distinct, and it therefore follows that

$$31 = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3.$$

Associated with these we have three other ideals deduced by the second theorem of last article; namely,

$$\mathfrak{p}_1' = \left[31,\ 31\theta,\ -\ 3-11\theta+\theta^2\right],$$
$$\mathfrak{p}_2' = \left[31,\ 31\theta,\ -13-24\theta+\theta^2\right],$$
$$\mathfrak{p}_3' = \left[31,\ 31\theta,\ -15-27\theta+\theta^2\right].$$

But these are not prime, as in the other case; in fact

$$\mathfrak{p}_1' = \mathfrak{p}_2\mathfrak{p}_3,\quad \mathfrak{p}_2' = \mathfrak{p}_3\mathfrak{p}_1,\quad \mathfrak{p}_3' = \mathfrak{p}_1\mathfrak{p}_2.$$

It will be sufficient to verify the last of these relations; the others may be proved in a similar way. Forming the nine products of pairs of elements of $\mathfrak{p}_1$ and $\mathfrak{p}_2$, these are

$$\begin{aligned}
31^2 &= 31\,.\,31,\\
31\,(11+\theta) &= 11\,.\,31+1\,.\,31\theta,\\
31\,(24+\theta) &= 24\,.\,31+1\,.\,31\theta,\\
(11+\theta)(24+\theta) &= (-15-27\theta+\theta^2)+2\,.\,31\theta+9\,.\,31,\\
(11+\theta)(13+\theta^2) &= 11\,(-15-27\theta+\theta^2)+10\,.\,31\theta+10\,.\,31,\\
31\,(3+\theta^2) &= 31\,(-15-27\theta+\theta^2)+27\,.\,31\theta+18\,.\,31,\\
31\,(13+\theta^2) &= 31\,(-15-27\theta+\theta^2)+27\,.\,31\theta+28\,.\,31,\\
(24+\theta)(3+\theta^2) &= 24\,(-15-27\theta+\theta^2)+21\,.\,31\theta+14\,.\,31,\\
(3+\theta^2)(13+\theta^2) &= 16\,(-15-27\theta+\theta^2)+14\,.\,31\theta+9\,.\,31.
\end{aligned}$$

Hence every number contained in $\mathfrak{p}_1\mathfrak{p}_2$ is contained in $\mathfrak{p}_3'$; that is, $\mathfrak{p}_1\mathfrak{p}_2$ is divisible by $\mathfrak{p}_3'$; and then, by a comparison of norms, we infer that $\mathfrak{p}_3' = \mathfrak{p}_1\mathfrak{p}_2$.

There is no difficulty in proving that these results are general. Thus, suppose p is any prime of the form $6n+1$, of which 2 is a cubic residue: then the congruence

$$a^3+2 \equiv 0,\ (\mathrm{mod}\ p),$$

has three incongruent roots, say

$$a \equiv a_1,\ a_2,\ a_3,\ (\mathrm{mod}\ p).$$

Let b_1, b_2, b_3 be the corresponding values of b, found from

$$b+a^2 \equiv 0,\ (\mathrm{mod}\ p);$$

and write
$$\mathfrak{p}_i = \left[p,\ a_i+\theta,\ b_i+\theta^2\right],$$
$$\mathfrak{p}_i' = \left[p,\ p\theta,\ -b_i-a_i\theta+\theta^2\right],\quad (i=1,2,3);$$

then the nine products formed from $\mathfrak{p}_1\mathfrak{p}_2$ are

$$p^2 = p\cdot p,\quad p(a_1+\theta) = p\theta + a_1\cdot p,\quad p(a_2+\theta) = p\theta + a_2\cdot p,$$

$$(a_1+\theta)(a_2+\theta) = (-b_3 - a_3\theta + \theta^2) + \frac{a_1+a_2+a_3}{p}\cdot p\theta + \frac{b_3+a_1a_2}{p}\cdot p,$$

$$(a_1+\theta)(b_2+\theta^2) = a_1(-b_3 - a_3\theta + \theta^2) + \frac{b_2+a_3a_1}{p}\cdot p\theta + \frac{a_1(b_2+b_3)+2}{p}\cdot p,$$

$$(a_2+\theta)(b_1+\theta^2) = a_2(-b_3 - a_3\theta + \theta^2) + \frac{b_1+a_2a_3}{p}\cdot p\theta + \frac{a_2(b_3+b_1)+2}{p}\cdot p,$$

$$p(b_2+\theta^2) = p(-b_3 - a_3\theta + \theta^2) + a_3\cdot p\theta + (b_2+b_3)\cdot p,$$

$$p(b_1+\theta^2) = p(-b_3 - a_3\theta + \theta^2) + a_3\cdot p\theta + (b_3+b_1)\cdot p,$$

$$(b_1+\theta^2)(b_2+\theta^2) = (b_1+b_2)(-b_3 - a_3\theta + \theta^2)$$
$$+\frac{a_3(b_1+b_2)+2}{p}\cdot p\theta + \frac{b_1b_2+b_2b_3+b_3b_1}{p}\cdot p.$$

In these relations, the quantities which appear in the form of fractions are really integral on account of the congruences satisfied by a and b.

Thus we have at once

$$a_1+a_2+a_3 \equiv 0, \pmod{p},$$

and $$b_2b_3+b_3b_1+b_1b_2 \equiv 2\left(\frac{1}{a_1a_3}+\frac{1}{a_3a_1}+\frac{1}{a_1a_2}\right) \equiv 0;$$

moreover $$a_3(b_3+a_1a_2) \equiv 2 + a_1a_2a_3 \equiv 0,$$

and hence $$b_3+a_1a_2 \equiv 0 \equiv b_1+a_2a_3 \equiv b_2+a_3a_1;$$

finally $$a_1(b_2+b_3)+2 \equiv a_1(b_2+b_3+b_1) \equiv -a_1\Sigma a^2 \equiv 0$$
$$\equiv a_2(b_3+b_1)+2 \equiv a_3(b_1+b_2)+2.$$

Consequently, as in the special case, it follows that

$$\mathfrak{p}_1' = \mathfrak{p}_2\mathfrak{p}_3,\quad \mathfrak{p}_2' = \mathfrak{p}_3\mathfrak{p}_1,\quad \mathfrak{p}_3' = \mathfrak{p}_1\mathfrak{p}_2.$$

4. It appears, then, that, leaving out of account the exceptional primes 2 and 3, all real primes may be arranged in three categories, namely, primes of the form $6n-1$, primes of the form $6n+1$ of which 2 is a cubic residue, and primes of the form $6n+1$ of which 2 is not a cubic residue. Primes of the first category may be expressed, in the complex theory we are considering, as the product of two prime factors (real or ideal); those of the second as the product of

three prime factors; while those of the third remain primes in the complex theory. There does not appear to be any advantage in distinguishing the resolution into real factors from that into ideal factors; of course, when any prime ideal which occurs can be reduced to the standard form of a principal ideal, there is a practical convenience in doing so; but this reduction is not essential, and does not affect any calculations into which the ideal enters, except in a merely formal way.

The resolution of any integer $\omega = x+y\theta+z\theta^2$ into its prime factors is effected by first calculating its norm; the real prime factors of this will correspond to the required factors of ω, and, since, by the methods explained, $N(\omega)$ can be resolved into its prime factors, it only remains to find out which of these are contained in ω. This can be done, as will now be shown, by a finite number of trials.

Let p be a prime of the first or second category which divides $N(\omega)$. Resolve p into its factors $\mathfrak{p}\mathfrak{p}'$, so that

$$N(\mathfrak{p}) = p, \quad N(\mathfrak{p}') = p^2;$$

and, with the notation already employed, suppose

$$\mathfrak{p} = \left[p,\ a+\theta,\ b+\theta^2\right],$$

$$\mathfrak{p}' = \left[p,\ p\theta,\ -b-a\theta+\theta^2\right].$$

Suppose that ω contains $\mathfrak{p}$, but not $\mathfrak{p}'$, as a factor; then, if μ is any real number contained in $\mathfrak{p}'$ but not in $\mathfrak{p}$, $\mathfrak{o}\mu$ is divisible by $\mathfrak{p}'$ but not by $\mathfrak{p}$, and therefore $\mathfrak{o}\mu\omega$ by $\mathfrak{p}\mathfrak{p}'$, that is, by $\mathfrak{o}p$, but not by p^2. Now

$$-b-a\theta+\theta^2 = (b+\theta^2)-a(a+\theta)+(a^2-2b),$$

and

$$a^2-2b \equiv 3a^2, \quad (\text{mod } p);$$

therefore a^2-2b is not a multiple of p, and we may put

$$\mu = -b-a\theta+\theta^2.$$

This gives

$$\begin{aligned}\mu\omega = \ & (-bx+2y-2az) \\ & +(-ax-by+2z)\,\theta \\ & +(x-ay-bz)\,\theta^2,\end{aligned}$$

and ω is divisible by $\mathfrak{p}$ if

$$\left.\begin{aligned} bx-2y+2az &\equiv 0, \\ ax+by-2z &\equiv 0, \\ x-ay-bz &\equiv 0, \end{aligned}\right\} \quad (\text{mod } p).$$

On account of the relation connecting a and b, it will be found that these three congruences reduce to the single one

$$x-ay-bz\equiv 0, \quad (\text{mod } p).$$

If this condition is satisfied, put

$$\frac{-bx+2y-2az}{p}=x',$$

$$\frac{-ax-by+2z}{p}=y',$$

$$\frac{x-ay-bz}{p}=z';$$

then
$$\mu\omega/p=x'+y'\theta+z\,\theta^2,$$

and this will again involve the factor $\mathfrak{p}$, if

$$x'-ay'-bz'\equiv 0, \quad (\text{mod } p).$$

Now $p\,(x'-ay'-bz')=(-2b+a^2)\,x+(2+2ab)\,y+(-4a+b^2)\,z$;

therefore ω is divisible by $\mathfrak{p}^2$ if

$$(-2b+a^2)\,x+(2+2ab)\,y+(-4a+b^2)\,z\equiv 0, \quad (\text{mod } p^2);$$

and, in the same way, we can express the condition that ω is divisible by $\mathfrak{p}^r$ in the form

$$l_r x+m_r y+n_r z\equiv 0, \quad (\text{mod } p^r).$$

In general ω involves the factor $\mathfrak{p}$ r times exactly if the above congruence is satisfied, while

$$l_{r+1}x+m_{r+1}y+n_{r+1}z\equiv 0, \quad (\text{mod } p^{r+1}),$$

is not satisfied.

In the same way, the number $\lambda=a+\theta$ is contained in $\mathfrak{p}$, but not in $\mathfrak{p}'$, and any number $\omega=x+y\theta+\theta^2$ will involve $\mathfrak{p}'$ as a factor if

$$\lambda\omega\equiv 0, \quad (\text{mod } p),$$

or, which is the same thing, if

$$\left.\begin{aligned} ax+2z&\equiv 0,\\ x+ay&\equiv 0,\\ y+az&\equiv 0,\end{aligned}\right\}\ (\text{mod } p).$$

Only two of these congruences are independent.

The repeated occurrence of $\mathfrak{p}'$ may be detected by finding the

highest value of r for which the congruence $\lambda^r \omega \equiv 0$, (mod p^r), is satisfied.

When p is a prime of the second category, so that $p = \mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3$, the criteria for the divisibility of $x + y\theta + z\theta^2$ by $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$, respectively, may be written

$$\left.\begin{aligned} x - a_1 y - b_1 z \equiv 0, \\ x - a_2 y - b_2 z \equiv 0, \\ x - a_3 y - b_3 z \equiv 0, \end{aligned}\right\} \pmod{p},$$

and as a verification it may be observed that if all these conditions are satisfied, it follows that

$$x \equiv y \equiv z \equiv 0, \pmod{p},$$

that is, $x + y\theta + z\theta^2$ is divisible by p.

Finally, if p is a prime of the third category, $x + y\theta + z\theta^2$ is divisible by p only if

$$x \equiv y \equiv z \equiv 0, \pmod{p}.$$

These congruential relations are analogous to those employed by Kummer in his papers on ideal primes.

On the Algebraical Integers derived from an Irreducible Cubic Equation. By G. B. Mathews.

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1. Let θ, θ', θ'' be the roots of the irreducible cubic

$$f(\theta) = \theta^3 + a\theta^2 + \beta\theta + \gamma = 0,$$

where a, β, γ are ordinary integers. Then there will be associated with it three conjugate *corpora* $\Omega(\theta)$, $\Omega(\theta')$, $\Omega(\theta'')$, the corpus $\Omega(\theta)$ being made up of the aggregate of all quantities such as

$$\omega = \frac{x + y\theta + z\theta^2}{t},$$

x, y, z, t being ordinary integers; and similarly for $\Omega(\theta')$, $\Omega(\theta'')$. (The conjugate corpora may or may not be distinct.)

In order that ω may be an algebraical integer it is necessary and sufficient that, ω', ω'' being conjugate to ω, the quantities

$$\omega+\omega'+\omega'', \quad \omega'\omega''+\omega''\omega+\omega\omega', \quad \omega\omega'\omega''$$

shall be rational integers; and this leads to three congruences which must be satisfied by x, y, z, namely,

$$3x-ay+(a^2-2\beta)\,z \equiv 0, \pmod{t} \quad \text{.................(1)},$$

$$3x^2+\beta y^2+(\beta^2-2\gamma a)\,z^2-(a\beta-3\gamma)yz+2(a^2-2\beta)zx-2axy \equiv 0, \pmod{t^2}$$
$$\text{...............(2)},$$

$$x^3+\gamma y^3+\gamma^2z^3-ax^2y+(a^2-2\beta)\,x^2z+\beta xy^2+(\beta^2-2\gamma a)\,xz^2-(a\beta-3\gamma)\,xyz$$
$$+a\gamma y^2z+\beta\gamma yz^2 \equiv 0, \pmod{t^3} \quad \text{..................(3)}.$$

2. It will be supposed, in the first instance, that t is prime to 3. Then the second and third congruences are not affected if they are multiplied by 3 and 27 respectively, and the moduli left unaltered; x may then be eliminated by putting

$$3x = \xi t+ay-(a^2-2\beta)\,z.$$

In order to express the results in the most convenient form, the following abbreviations will be used:—

$$\left.\begin{array}{c} u = y-az \\ A = 3\beta-a^2, \quad B = 9\gamma-a\beta, \quad C = 3a\gamma-\beta^2 \\ P = 2a^3-9a\beta+27\gamma, \quad Q = a^2\beta+9a\gamma-6\beta^2 \\ R = 6a^2\gamma-a\beta^2-9\beta\gamma, \quad S = 9a\beta\gamma-2\beta^3-27\gamma^2 \\ D = \tfrac{1}{3}(B^2-4AC) = 27\gamma^2+4\beta^3+4a^3\gamma-a^2\beta^2-18a\beta\gamma \end{array}\right\} \text{......(4)}.$$

D may be called the discriminant of $f(\theta)$, and it will be observed that $(A, B, C \between \theta, 1)^2$ is the Hessian, and $(P, Q, R, S \between \theta, 1)^3$ the cubicovariant of $f(\theta)$, each with its coefficients absolutely determined.

The congruence (1) may now be written

$$3x-au-2\beta z \equiv 0, \pmod{t} \quad \text{.................(5)},$$

and if we substitute $au+2\beta z+\xi t$ for $3x$ in (2), after multiplying by 3, it will be found that the term in t on the left hand vanishes identically, and that (2) is replaced by the equivalent congruence

$$Au^2+Buz+Cz^2 \equiv 0, \pmod{t^2} \quad \text{..............(6)}.$$

When the same substitution is made in (3), after multiplying by 27, the coefficient of t^2 vanishes identically; the coefficient of t may be reduced to the form $3\xi t\,(Au^2+Buz+Cz^2)$, which, on account of the congruence last written, is a multiple of t^3, and may therefore be omitted; and finally, after some algebraical reductions, it is found that (3) may be replaced by

$$Pu^3+3Qu^2z+3Ruz^2+Sz^3 \equiv 0, \quad (\text{mod } t^3) \text{ ....(7).}$$

3. It is now necessary to discuss the simultaneous congruences (6) and (7). Unfortunately, this is a matter of considerable difficulty, because the algebraical theory of elimination cannot be applied except under certain limitations. By following the analogy of Sylvester's dialytic method of elimination, we may establish the theorem that, if R is the algebraical resultant of two binary quantics $F(x, y)$, $G(x, y)$, then, in order that the simultaneous congruences

$$F(x, y) \equiv 0, \quad G(x, y) \equiv 0, \quad (\text{mod } m),$$

may admit of solutions such that no common factor of x and y may divide m, it is *necessary* that $R \equiv 0$, (mod m).

Now in the congruences (6) and (7), we may suppose that u and z have no common factor which divides t; for, if they had, it would follow from (5) and (4) that x, y, z, t would all have a common factor, and hence $(x+y\theta+z\theta^2)/t$ would not be in its lowest terms. Therefore supposing, as we may do, that ω is in its lowest terms, and observing that the resultant of $(A, B, C \between u, z)^2$ and $(P, Q, R, S \between u, z)^3$ is $27D^3$, we infer from (6) and (7) that

$$27D^3 \equiv 0, \quad (\text{mod } t^2) \text{........................(8).}$$

If t is not prime to 3, we must write instead of (6) and (7),

$$Au^2+Buz+Cz^2 \equiv 0, \quad (\text{mod } 3t^2) \text{ ....(9),}$$

$$Pu^3+3Qu^2z+3Ruz^2+Sz^3 \equiv 0, \quad (\text{mod } 27t^3) \text{......(10),}$$

and consequently, instead of (8),

$$27D^3 \equiv 0, \quad (\text{mod } 3t^2).$$

Both cases are included in

$$9D^3 \equiv 0, \quad (\text{mod } t^2) \text{ ....(11),}$$

and the conclusion is that the investigation may be confined to those integers t the squares of which divide $9D^3$.

4. There are so many different cases to consider, according to the distribution of the prime factors of A, B, C, P, Q, R, S, and D, that it seems best not to attempt a complete enumeration. The congruences (6) and (7) or (9) and (10) must, of course, be fully discussed in any particular case that may arise; there is no difficulty in doing this when the function $f(\theta)$ has once been chosen.

There are one or two specially interesting cases which deserve attention, and will serve to illustrate the theory.

The first of these is when the congruences (6) and (7) are satisfied *identically*; that is to say, when

$$A \equiv B \equiv C \equiv 0, \quad (\text{mod } t^2) \quad \ldots\ldots\ldots\ldots\ldots (12),$$

$$P \equiv Q \equiv R \equiv S \equiv 0, \quad (\text{mod } t^3).$$

These are equivalent to two distinct conditions, which may be expressed by

$$A \equiv 0, \quad (\text{mod } t^2),$$

$$P \equiv 0, \quad (\text{mod } t^3) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13);$$

and it will be found that D is divisible by t^6. The integers u and z may be chosen at pleasure, and then $y = u + az$, and x, y, z are connected by the single relation

$$3x - ay + (a^2 - 2\beta) z \equiv 0, \quad (\text{mod } t);$$

or, say,

$$x \equiv \lambda y + \mu z, \quad (\text{mod } t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

where λ, μ are determinate to modulus t. Suppose they have their least positive values, then the general form of ω for this value of t is

$$\omega = \frac{(\lambda + \theta)\, y + (\mu + \theta^2)\, z}{t} + hy + kz,$$

where h, k are rational integers. The essentially new integers thus introduced are

$$\omega_1 = \frac{\lambda + \theta}{t}, \quad \omega_2 = \frac{\mu + \theta^2}{t};$$

and it may be observed that, since

$$\theta = t\omega_1 - \lambda, \quad \theta^2 = t\omega_2 - \mu,$$

the modulus $[1, \omega_1, \omega_2]$ includes all the elements of the modulus $[1, \theta, \theta^2]$, and besides these a portion of the remaining integers in $\Omega(\theta)$.

In order to construct a case of this kind, suppose $\alpha = 1$, $t = 5$; then the congruences $A \equiv 0$, (mod 25), $P \equiv 0$, (mod 125), give

$$3\beta - 1 \equiv 0, \quad (\text{mod } 25),$$

$$27\gamma - 9\beta + 2 \equiv 0, \quad (\text{mod } 125).$$

If we take $\beta = 17$, the first congruence is satisfied, and the second leads to $\gamma \equiv 38$, (mod 125). It will be found that $\gamma = 38$ leads to a reducible equation in θ; but if we take $\gamma = 163$, we have the irreducible equation

$$\theta^3 + \theta^2 + 17\theta + 163 = 0,$$

for which $A = 2 \,.\, 5^2$, $B = 58 \,.\, 5^2$, $C = 8 \,.\, 5^2$,

$$P = 34 \,.\, 5^3, \quad Q = -2 \,.\, 5^3, \quad R = -194 \,.\, 5^3, \quad S = -5618 \,.\, 5^3,$$

$$D = 11 \,.\, 2^2 \,.\, 5^6.$$

The congruence (5) reduces to

$$x \equiv 2y + z, \quad (\text{mod } 5),$$

so that

$$\omega = \frac{(2+\theta)\, y + (1+\theta^2)\, z}{5}$$

is an integer when y, z are rational integers; and, as a verification, it will be found that if we put

$$\omega_1 = \frac{2+\theta}{5}, \quad \omega_2 = \frac{1+\theta^2}{5},$$

these quantities satisfy the equations

$$\omega_1^3 - \omega_1^2 + \omega_1 + 1 = 0,$$

$$\omega_2^3 + 6\omega_2^2 - 4\omega_2 - 212 = 0.$$

It is noticeable that ω_1 is an algebraical unit; and also that ω_2 is an integral function of ω_1, because

$$5\omega_2 - 1 = (5\omega_1 - 2)^2,$$

and hence

$$\omega_2 = 5\omega_1^2 - 4\omega_1 + 1.$$

5. The latter circumstance is not accidental; for if, as above, we put

$$\omega_1 = \frac{\lambda+\theta}{t}, \quad \omega_2 = \frac{\mu+\theta^2}{t},$$

we have
$$(t\omega_1-\lambda)^2 = t\omega_2-\mu,$$
and
$$\omega_2 = t\omega_1^2-2\lambda\omega_1+\frac{\lambda^2+\mu}{t};$$
now
$$3\lambda \equiv a, \quad (\text{mod } t),$$
$$3\mu \equiv -a^2+2\beta;$$
and therefore
$$9(\lambda^2+\mu) \equiv (6\beta-2a^2)$$
$$\equiv 2A \equiv 0, \quad (\text{mod } t);$$

that is, $(\lambda^2+\mu)/t$ is an integer, and ω_2 is an integral function of ω_1. Hence it may be inferred that the corpora $\Omega(\theta)$ and $\Omega(\omega_1)$ are identical in content; but the discriminant of $\Omega(\omega_1)$ is D/t^6 instead of D.

Thus, for the equation
$$\theta^3-\theta^2+\theta+1 = 0 \quad \text{.........................(15)}$$
(where θ has been written instead of ω_1),
$$A = 2, \quad B = 10, \quad C = -4,$$
$$D = 44.$$

Since D is divisible by the square of 2, we might expect to find integers of the form $(x+y\theta+z\theta^2)/2$; however, such integers do not exist, for the auxiliary congruences (6), (7) in this case are, on reduction,
$$u^2+uz \equiv 0, \quad (\text{mod } 2),$$
$$u^3-u^2z+uz^2+z^3 \equiv 0, \quad (\text{mod } 4),$$
and these can only be satisfied simultaneously if $u \equiv z \equiv 0$, (mod 2), leading to $x \equiv y \equiv z \equiv 0$, (mod 2).

It may be proved without difficulty that there are no integers of the form $(x+y\theta+z\theta^2)/11$, and hence that the equation (15) defines what may be called a primitive corpus $\Omega(\theta)$, that is, one which contains no integers except those of the form $x+y\theta+z\theta^2$ with x, y, z integral.

It may be worth while to observe that, since
$$4(Ax^2+Bx+C)^3+(Px^3+3Qx^2+3Rx+S)^2$$
$$= 27D(x^3+ax^2+\beta x+\gamma)^2 \text{.........(16)}$$

identically, it follows that, whenever A, B, C are divisible by t^2, and D is divisible by t^6,

$$Px^3+3Qx^2+3Rx+S \equiv 0, \quad (\text{mod } t^3),$$

identically; so that it is unnecessary to calculate the values of P, Q, R, S.

6. Another interesting case is when the congruence (6) is satisfied identically, but not (7). It follows from the identities

$$P = -2\alpha A+3B, \quad Q = -2\beta A+\alpha B,$$
$$R = -\beta B+2\alpha C, \quad S = -3\gamma B+2\beta C,$$

that P, Q, R, S are all divisible by t^2, but not by t^3. Now multiply (7) by P^2, and change the modulus to t^7; then the new congruence, equivalent to the old one, is

$$(Pu+Qz)^3-3\,(Q^2-PR)(Pu+Qz)\,z^2$$
$$+(P^2S-3PQR+2Q^3)\,z^3 \equiv 0, \quad (\text{mod } t^7).$$

But Q^2-PR is a multiple of DA, and therefore of t^6; hence the second term is divisible by t^8. Also $P^2S-3PQR+2Q^3$ is divisible by D^2, and therefore by t^8, so that the congruence reduces to

$$(Pu+Qz)^3 \equiv 0, \quad (\text{mod } t^7),$$

whence
$$\left(\frac{P}{t^2}u+\frac{Q}{t^2}z\right)^3 \equiv 0, \quad (\text{mod } t);$$

and, if t is not divisible by any cube, this leads to

$$Pu+Qz \equiv 0, \quad (\text{mod } t^3).$$

[If t is divisible by a cube, some modification is necessary; but, as already said, it seems best to omit these discussions of detail.]

The solution of this congruence will be of the form

$$u \equiv kz, \quad (\text{mod } t),$$

where k is determinate to modulus t; and therefore it will follow that

$$y \equiv qz, \quad x \equiv pz, \quad (\text{mod } t),$$

where p, q may be taken to be determinate numbers between 0 and t; so that, instead of obtaining, as in the last case, *two* integers ω_1 and ω_2 (or ω_1 and ω_1^2), of which the form is fractional, we shall have only one, namely,

$$\omega_1 = \frac{p+q\theta+\theta^2}{t}.$$

Of the conditions
$$A = 3\beta - \alpha^2 \equiv 0, \quad (\text{mod } t^2),$$
$$B = 9\gamma - \alpha\beta \equiv 0,$$
$$C = 3\alpha\gamma - \beta^2 \equiv 0,$$
only two are independent, because
$$\beta A - \alpha B + 3C = 0$$
identically, so that if $A \equiv 0$ and $B \equiv 0$, it follows that $C \equiv 0$, always supposing that t is prime to 3.

If t and α are assigned, suitable values of β and γ may be found from
$$3\beta \equiv \alpha^2, \quad (\text{mod } t^2), \qquad 27\gamma \equiv \alpha^3, \quad (\text{mod } t^2).$$

For instance, if $\alpha = 1$, $t = 5$, it will be found that $\beta \equiv 17$, $\gamma \equiv 13$, (mod 25); and, if we take θ to be a root of
$$\theta^3 + \theta^2 + 17\theta + 13 = 0,$$
which is irreducible,
$$A = 50, \quad B = 100, \quad C = -250, \quad D = 32 \,.\, 5^4,$$
$$P = 8 \,.\, 25, \quad Q = -64 \,.\, 25, \quad R = -88 \,.\, 25, \quad S = -496 \,.\, 25\,;$$
and the congruence (7) is
$$8\,(u^3 - 24u^2z - 33uz^2 - 62z^3) \equiv 0, \quad (\text{mod } 5),$$
that is,
$$8\,(u - 3z)^3 \equiv 0, \quad (\text{mod } 5)\,;$$
therefore
$$u \equiv 3z, \quad y \equiv 4z, \quad x \equiv 4z, \quad (\text{mod } 5),$$
and all the integers of the set here considered are rational multiples of
$$\omega_1 = \frac{4 + 4\theta + \theta^2}{5}.$$

It will be found that ω_1 satisfies the equation
$$\omega_1^3 + 5\omega_1^2 + 15\omega_1 - 5 = 0.$$

The relation between θ and ω_1 may be expressed in the forms
$$\left.\begin{aligned} \omega_1^2 &= -3 - 4\theta, \\ 5\omega_1 &= 4 + 4\theta + \theta^2, \end{aligned}\right\} \qquad \left.\begin{aligned} \theta^2 &= \omega_1^2 + 5\omega_1 - 7, \\ 4\theta &= -\omega_1^2 - 3. \end{aligned}\right\}$$

For the equation satisfied by ω_1, we have
$$A = 20, \quad B = -120, \quad C = -300, \quad D = 2^9 \,.\, 5^2\,;$$

and, since A, B, C are divisible by 2^2, and D by 2^6, we shall have integers of the form

$$\eta = \frac{x+y\omega_1+z\omega_1^2}{2}.$$

Proceeding as already explained, it will be found that

$$\eta_1 = \frac{1+\omega_1}{2}, \quad \eta_2 = \frac{1+\omega_1^2}{2}$$

are integers; that $\qquad \eta_2 = 2\eta_1^2 - 2\eta_1 + 1;$

and that η_1 is a root of the equation

$$\eta^3+\eta^2+2\eta-2 = 0,$$

for which $\qquad A = 5, \quad B = -20, \quad C = -10,$

$$D = 200 = 2^3 . 5^2;$$

$$P = -70, \quad Q = -40, \quad R = 20, \quad S = -160.$$

On examining the auxiliary congruences (6) and (7) for $t = 2, 3, 5$, respectively, it will be found that they do not admit of solutions distinct from $u \equiv z \equiv 0$; hence $\Omega(\eta)$ is a primitive corpus.

It will be found that, writing η for η_1,

$$\omega_1 = 2\eta - 1,$$

$$\theta = -\eta^2+\eta-1;$$

and these may be regarded as Tchirnhausen transformations by which the equations in ω_1 and θ may be derived from the equation in η. It may, I think, be inferred that every corpus that is not primitive may be derived from a primitive corpus by a transformation of this kind.

7. There is one other special case of the general theory which is of some practical importance. It may happen that D is divisible by t^2, while A and P are prime to t. In this case the congruences (6) and (7) are satisfied by putting

$$2Au+Bz \equiv 0, \quad (\text{mod } t),$$

$$Pu+Qz \equiv 0, \quad (\text{mod } t),$$

which are consistent, because

$$2AQ-BP = -9D \equiv 0, \quad (\text{mod } t^2).$$

(It is supposed that t is odd and prime to 3.)

As an example of this case, take

$$\theta^3+55\theta^2+67\theta+77=0,$$

which occurs in connexion with complex multiplication of elliptic functions for $\Delta=53$ (*Proc. Lond. Math. Soc.*, Vol. XXI., p. 217).

Here $\quad A=-8.353,\quad B=-8.374,\quad C=8.1027,$

$$D=2^{10}.5^4.53,$$

while P is prime to 5. Hence, putting $t=25$, we find, after some reductions, that the auxiliary congruences give

$$x\equiv 13z,\quad y\equiv z,\quad (\text{mod } 25),$$

and the quantity $\qquad \eta=\dfrac{13+\theta+\theta^2}{25}$

satisfies the equation $\eta^3-115\eta^2+107\eta-25=0$.

For this equation

$$A=-8.1613,\quad B=8.1510,\quad C=-8.353,$$

$$D=2^{10}.53.$$

Now A, B, C are all divisible by 2^3, and D is divisible by 2^6; hence we find that

$$\omega=\frac{x+y\eta+z\eta^2}{2}$$

is integral if $\qquad x\equiv y+z,\quad (\text{mod } 2).$

Putting, then, $\qquad \zeta=\dfrac{1+\eta}{2},$

we find that $\qquad \zeta^3-59\zeta^2+85\zeta-31=0,$

an equation with

$$A=-2.1613,\quad B=4.1184,\quad C=-2.869,$$

$$D=2^4.53.$$

It will be found on trial that the corpus $\Omega(\zeta)$ is primitive, and that

$$\eta=2\zeta-1,$$

$$\theta=4\zeta^2-234\zeta+169.$$

On the Expansion of some Infinite Products. By Prof. L. J. Rogers. Received June 5th, 1893. Read June 8th, 1893.

1. It is a well-known theorem that, if $q < 1$, then

$$1/(1-\lambda)(1-\lambda q)(1-\lambda q^2)\ldots = 1+\frac{\lambda}{1-q}+\frac{\lambda^2}{(1-q)(1-q^2)}+\ldots \quad \ldots(1).$$

It will be found convenient to use the symbol (λ) for the infinite product $(1-\lambda)(1-\lambda q)(1-\lambda q^2)\ldots$, and to write the above equation in the form

$$1/(\lambda) = 1+\Sigma\frac{\lambda^r}{(1-q^r)!},$$

where r is to receive all positive integral values, and where $(1-q^r)!$ denotes the product $(1-q)(1-q^2)\ldots(1-q^r)$.

The following abbreviations will also be used in the following pages:—

$H_r(\lambda_1, \lambda_2, \lambda_3, \ldots)$ will denote the coefficient of x^r in

$$1/(\lambda_1 x)(\lambda_2 x)(\lambda_3 x)\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

while $h_r(\lambda_1, \lambda_2, \ldots)$ will be used for $H_r(\lambda_1, \lambda_2, \ldots)(1-q^r)!$ Moreover $H_r(\mu_1, \mu_2, \ldots/\lambda_1, \lambda_2, \ldots)$ will be written for the coefficient of x^r in

$$(\mu_1 x)(\mu_2 x)\ldots \div (\lambda_1 x)(\lambda_2 x)\ldots,$$

while $h_r(\mu_1, \mu_2\ldots/\lambda_1, \lambda_2, \ldots)$ will $= (1-q^r)!\, H_r(\mu_1, \mu_2, \ldots/\lambda_1, \lambda_2, \ldots)$.

Thus
$$1+\Sigma x^r H_r(\lambda, \mu) = 1/(\lambda x)(\mu x)$$

$$= \left\{1+\frac{\lambda x}{1-q}+\frac{\lambda^2 x^2}{(1-q)(1-q^2)}+\ldots\right\}\left\{1+\frac{\mu x}{1-q}+\frac{\mu^2 x^2}{(1-q)(1-q^2)}+\ldots\right\},$$

whence

$$h_r(\lambda, \mu) = \lambda^n+\frac{1-q^n}{1-q}\lambda^{n-1}\mu+\frac{(1-q^n)(1-q^{n-1})}{(1-q)(1-q^2)}\lambda^{n-2}\mu^2+\ldots\mu^n \quad \ldots(3),$$

a series having a certain resemblance to the ordinary binomial expansion.

The symbols of operation δ, η with respect to any quantity λ will be defined by the equations

$$\delta f(\lambda) \equiv \frac{f(\lambda)-f(\lambda q)}{\lambda} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$\eta f(\lambda) \equiv f(\lambda q).$$

Thus
$$\delta\lambda^r = \lambda^{r-1}(1-q^r),$$
$$\delta^2\lambda^r = \lambda^{r-2}(1-q^r)(1-q^{r-1}), \text{ \&c.}$$

The symbols δ_m, η_m will be used in the above sense with reference to the quantity λ_m, so that
$$\eta_1 f(\lambda_1) = f(\lambda_1 q), \text{ \&c.}$$

It is easy to see then that the symbol $\dfrac{1}{(\lambda\delta_1)}$, *i.e.*
$$1+\frac{\lambda\delta_1}{1-q}+\frac{\lambda^2\delta_1}{(1-q)(1-q^2)}+\ldots,$$
operating on λ_1^r, gives
$$\lambda_1^r+\frac{1-q^r}{1-q}\lambda\lambda_1^{r-1}+\frac{(1-q^r)(1-q^{r-1})}{(1-q)(1-q^2)}\lambda^2\lambda_1^{r-2}+\ldots = h_r(\lambda, \lambda_1), \text{ by (3).}$$

Hence
$$\frac{1}{(\lambda\delta_1)}\cdot\frac{1}{(\lambda_1\lambda_2)} = 1+\Sigma\lambda_2^r H_r(\lambda, \lambda_1) = \frac{1}{(\lambda\lambda_2)(\lambda_1\lambda_2)} \quad \ldots\ldots\ldots(5).$$

We may now establish certain lemmata which will be useful hereafter for reference.

Lemma I.—If K_0, K_1 ... are independent of λ_1, then
$$\frac{1}{(\lambda\delta_1)}(K_0+K_1\lambda_1+K_2\lambda_1^2+\ldots) = K_0+K_1h_1(\lambda, \lambda_1)+K_2h_2(\lambda, \lambda_1)+\ldots.$$

This follows obviously from what has gone before.

Lemma II.—The operation $(\lambda\delta_1)$ will be equivalent to
$$\left(\frac{\lambda}{\lambda_1}\right)\frac{1}{\left(\dfrac{\lambda}{\lambda_1}\eta_1\right)},$$
i.e.
$$\left(\frac{\lambda}{\lambda_1}\right)\left\{1-\frac{\lambda}{\lambda_1}\cdot\frac{1}{1-q}\eta_1+\frac{\lambda^2}{\lambda_1^2}\cdot\frac{1}{(1-q)(1-q^2)}\eta_1^2+\ldots\right\}.$$

For
$$\delta_1 = \frac{1-\eta_1}{\lambda_1},$$
$$\delta_1^2 = \frac{1-\eta_1}{\lambda_1^2}-\frac{1-\eta_1}{\lambda_1^2 q}\eta_1 = \frac{(1-\eta_1)(q-\eta_1)}{\lambda_1^2 q},$$
$$\delta_1^3 = \frac{(1-\eta_1)(q-\eta_1)(q^2-\eta_1)}{\lambda_1^3 q^3},$$

and $$\delta_1^r = \frac{(1-\eta_1)(q-\eta_1)\ldots(q^{r-1}-\eta_1)}{\lambda_1^r q^{\frac{1}{2}r(r-1)}}.$$

Now it is well known that

$$(x) = 1 - \frac{x}{1-q} + \frac{x^2q}{(1-q)(1-q^2)} - \frac{x^3q^3}{(1-q)(1-q^2)(1-q^3)} + \ldots;$$

therefore

$$(\lambda\delta_1) = 1 + \frac{\lambda}{\lambda_1}\cdot\frac{\eta_1-1}{1-q} + \frac{\lambda^2}{\lambda_1^2}\cdot\frac{(\eta_1-1)(\eta_1-q)}{(1-q)(1-q^2)} + \ldots.$$

But it is a known theorem that

$$\frac{(ax)}{(x)} = 1 + \frac{1-a}{1-q}x + \frac{(1-a)(1-aq)}{(1-q)(1-q^2)}x^2 + \ldots \quad \ldots\ldots\ldots(6);$$

therefore $$(\lambda\delta_1) = \left(\frac{\lambda}{\lambda_1}\right)\frac{1}{\left(\frac{\lambda}{\lambda_1}\eta_1\right)},$$

if we remember that η_1 is not to operate on the coefficients involving $\frac{\lambda}{\lambda_1}$, *i.e.* $\left(\frac{\lambda}{\lambda_1}\right)$, and the powers of $\left(\frac{\lambda}{\lambda_1}\right)$ always precede the operations η_1, η_1^2, ... in the expansion.

Lemma III. $\frac{1}{(\lambda\delta_1)}(\lambda_1-\lambda)\,\psi(\lambda_1)$ will $= \lambda_1\frac{1}{(\lambda q\delta_1)}\,\psi(\lambda_1)$.

For $$(\lambda\delta_1)\,\lambda_1 f(\lambda_1)$$

$$= \left(\frac{\lambda}{\lambda_1}\right)\left\{1 + \frac{\lambda}{\lambda_1}\cdot\frac{\eta_1}{1-q} + \frac{\lambda^2}{\lambda_1^2}\cdot\frac{\eta_1^2}{(1-q)(1-q^2)} + \ldots\right\}\lambda_1 f(\lambda_1)$$

(by Lemma II.)

$$= \lambda_1\left(\frac{\lambda}{\lambda_1}\right)\left\{1 + \frac{\lambda}{\lambda_1}\cdot\frac{q\eta_1}{1-q} + \frac{\lambda^2}{\lambda_1^2}\cdot\frac{q^2\eta_1^2}{(1-q)(1-q^2)} + \ldots\right\}f(\lambda_1)$$

$$= \lambda_1\left(\frac{\lambda}{\lambda_1}\right)\frac{1}{\left(\frac{\lambda q}{\lambda_1}\eta_1\right)}f(\lambda_1)$$

$$= (\lambda_1-\lambda)(\lambda q\delta_1)\,f(\lambda_1) \quad \text{(by Lemma II.)}.$$

Let $$\psi(\lambda_1) = (\lambda q \delta_1) f(\lambda_1),$$

so that $$f(\lambda_1) = \frac{1}{(\lambda q \delta_1)} \psi(\lambda_1),$$

then $$(\lambda_1 - \lambda)\, \psi(\lambda_1) = (\lambda \delta_1)\, \lambda_1 f(\lambda_1),$$

i.e., $$\frac{1}{(\lambda \delta_1)} (\lambda_1 - \lambda)\, \psi(\lambda_1) = \lambda_1 \frac{1}{(\lambda q \delta_1)} \psi(\lambda_1).$$

Similarly $$\frac{1}{(\lambda \delta_1)} (\lambda_1 - \lambda)(\lambda_1 - \lambda q)\, \psi(\lambda_1) = \lambda_1 \frac{1}{(\lambda q \delta_1)} (\lambda_1 - \lambda q)\, \psi(\lambda_1)$$
$$= \lambda_1^2 \frac{1}{(\lambda q^2 \delta_1)} \psi(\lambda_1), \text{ \&c.}$$

Lemma IV. $$\frac{1}{(\lambda \delta_1)} \cdot \frac{\psi(\lambda_1)}{(\lambda_1 \mu)} = \frac{1}{(\lambda_1 \mu)} \cdot \frac{1}{(\lambda \mu \eta_1)} \cdot \frac{1}{(\lambda \delta_1)} \psi(\lambda_1),$$

where μ is independent of λ_1.

Since $$\frac{(\lambda \mu)}{(\lambda_1 \mu)} = 1 + \frac{\lambda_1 - \lambda}{1-q} \mu + \frac{(\lambda_1 - \lambda)(\lambda_1 - \lambda q)}{(1-q)(1-q^2)} \mu^2 + \dots;$$

therefore $$\frac{1}{(\lambda \delta_1)} \cdot \frac{\psi(\lambda_1)}{(\lambda_1 \mu)}$$

$$= \frac{1}{(\lambda \mu)} \cdot \frac{1}{(\lambda \delta_1)} \left\{ 1 + \frac{\lambda_1 - \lambda}{1-q} \mu + \dots \right\} \psi(\lambda_1)$$

$$= \frac{1}{(\lambda \mu)} \left\{ \frac{1}{(\lambda \delta_1)} + \frac{\lambda_1 \mu}{1-q} \cdot \frac{1}{(\lambda q \delta_1)} + \frac{\lambda_1^2 \mu^2}{(1-q)(1-q^2)} \cdot \frac{1}{(\lambda q^2 \delta_1)} + \dots \right\} \psi$$

(by Lemma III.)

$$= \frac{1}{(\lambda \mu)} \left\{ 1 + \frac{\lambda_1 \mu (1 - \lambda \delta_1)}{1-q} + \frac{\lambda_1^2 \mu^2 (1 - \lambda \delta_1)(1 - \lambda q \delta_1)}{(1-q)(1-q^2)} + \dots \right\} \frac{1}{(\lambda \delta_1)} \psi$$

$$= \frac{1}{(\lambda \mu)} \cdot \frac{1}{(\lambda_1 \mu)} (\lambda \mu \lambda_1 \delta_1) \frac{1}{(\lambda \delta_1)} \psi$$

$$= \frac{1}{(\lambda_1 \mu)} \cdot \frac{1}{(\lambda \mu \eta_1)} \cdot \frac{1}{(\lambda \delta_1)} \psi(\lambda_1), \quad \text{(by Lemma II.).}$$

2. Let us now find the value of

$$\frac{1}{(\lambda \delta_1)} \cdot \frac{1}{(\lambda_1 \lambda_2)(\lambda_1 \lambda_3)},$$

which, when expanded as a series, gives

$$\frac{1}{(\lambda\delta_1)}\{1+\Sigma\lambda_1^r H_r(\lambda_2, \lambda_3)\}$$

$$= 1+\Sigma H_r(\lambda_2, \lambda_3)\, h_r(\lambda, \lambda_1), \quad \text{(by Lemma I.)}.$$

Now, by Lemma IV.,

$$\frac{1}{(\lambda\delta_1)}\cdot\frac{1}{(\lambda_1\lambda_2)(\lambda_1\lambda_3)}$$

$$=\frac{1}{(\lambda_1\lambda_2)}\cdot\frac{1}{(\lambda\lambda_2\eta_1)}\cdot\frac{1}{(\lambda\delta_1)}\cdot\frac{1}{(\lambda_1\lambda_3)}$$

$$=\frac{1}{(\lambda_1\lambda_2)}\cdot\frac{1}{(\lambda\lambda_2\eta_1)}\cdot\frac{1}{(\lambda\lambda_3)(\lambda_1\lambda_3)} \qquad \text{[by § 1, (5)]}$$

$$=\frac{1}{(\lambda\lambda_3)(\lambda_1\lambda_2)(\lambda_1\lambda_3)}\left\{1+\frac{1-\lambda_1\lambda_3}{1-q}\lambda\lambda_2+\frac{(1-\lambda_1\lambda_3)(1-\lambda_1\lambda_3 q)}{(1-q)(1-q^2)}\lambda^2\lambda_2^2+\dots\right\}$$

$$=\frac{(\lambda\lambda_1\lambda_2\lambda_3)}{(\lambda\lambda_2)(\lambda\lambda_3)(\lambda_1\lambda_2)(\lambda_1\lambda_3)} \qquad \text{[by § 1, (6)]}.$$

Hence

$$\frac{(\lambda\lambda_1\lambda_2\lambda_3)}{(\lambda\lambda_2)(\lambda\lambda_3)(\lambda_1\lambda_2)(\lambda_1\lambda_3)} = 1+H_1(\lambda_2, \lambda_3)h_1(\lambda, \lambda_1)+H_2(\lambda_2, \lambda_3)\,h_2(\lambda, \lambda_1)+\dots \qquad \text{(1)}.$$

If in this identity we write

$$\lambda = xe^{\theta i}, \quad \lambda_1 = xe^{-\theta i}, \quad \lambda_2 = e^{\phi i}, \quad \lambda_3 = e^{-\phi i},$$

we easily obtain 2 (1), on p. 176 of this volume.

3. Let us consider now the result of evaluating

$$\frac{1}{(\lambda\delta_1)}\cdot\frac{1}{(\lambda_1\lambda_2)(\lambda_1\lambda_3)(\lambda_1\lambda_4)},$$

which, by Lemma I.,

$$\equiv 1+\Sigma H_r(\lambda_2, \lambda_3, \lambda_4)\, h_r(\lambda, \lambda_1), \quad \text{see § 1 (2)}.$$

By Lemma IV., we get

$$\frac{1}{(\lambda_1\lambda_2)}\cdot\frac{1}{(\lambda\lambda_2\eta_1)}\cdot\frac{1}{(\lambda_1\lambda_3)}\cdot\frac{1}{(\lambda\lambda_3\eta_1)}\cdot\frac{1}{(\lambda\delta_1)}\cdot\frac{1}{(\lambda_1\lambda_4)}$$

$$=\frac{1}{(\lambda_1\lambda_2)}\cdot\frac{1}{(\lambda\lambda_2\eta_1)}\cdot\frac{(\lambda\lambda_1\lambda_3\lambda_4)}{(\lambda_1\lambda_3)(\lambda_1\lambda_4)(\lambda\lambda_3)},\text{ as in the preceding section,}$$

$$=\frac{1}{(\lambda\lambda_3)(\lambda_1\lambda_2)}\left\{1+\frac{\lambda\lambda_2}{1-q}\eta_1+\dots\right\}\frac{(\lambda\lambda_1\lambda_3\lambda_4)}{(\lambda_1\lambda_3)(\lambda_1\lambda_4)}$$

$$=\frac{(\lambda\lambda_1\lambda_3\lambda_4)}{(\lambda\lambda_3)(\lambda_1\lambda_2)(\lambda_1\lambda_3)(\lambda_1\lambda_4)}\left\{1+\frac{(1-\lambda_1\lambda_3)(1-\lambda_1\lambda_4)}{(1-q)(1-\lambda\lambda_1\lambda_3\lambda_4)}\lambda\lambda_2+\dots\right\}.$$

This last series is usually called a Heinean series, and has been very fully discussed in Heine's *Kugelfunctionen*, under the symbol

$$\phi\left\{\lambda_1\lambda_3,\ \lambda_1\lambda_4,\ \lambda\lambda_1\lambda_3\lambda_4,\ q,\ \lambda\lambda_2\right\},$$

Heine there shows that

$$\phi\left\{a,\ b,\ c,\ q,\ x\right\}=\frac{(ax)(b)}{(x)(c)}\,\phi\left\{\frac{c}{b},\ x,\ ax,\ q,\ b\right\},$$

and, by transforming the latter series by continual application of the same formula, he obtains a large series of equivalent expressions.

They may all be easily proved by considering the symmetry existing among λ_2, λ_3, λ_4 in the above expression, and by the symmetry in λ and λ_1, but he does not establish the connection between

$$\phi\left\{\lambda_1\lambda_3,\ \lambda_1\lambda_4,\ \lambda\lambda_1\lambda_3\lambda_4,\ q,\ \lambda\lambda_2\right\}$$

and $$1+\Sigma H_r(\lambda_2,\lambda_3,\lambda_4)\,h_r(\lambda,\lambda_1).$$

By writing $\lambda=e^{\theta i}$, $\lambda_1=e^{-\theta i}$, we may write the above relation in the form

$$\phi\left\{\lambda_3e^{-\theta i},\ \lambda_4e^{-\theta i},\ \lambda_3\lambda_4,\ q,\ \lambda_2e^{\theta i}\right\}(\lambda_2e^{\theta i})(\lambda_3\lambda_4)$$

$$=\frac{1}{P(\lambda_2)\,P(\lambda_3)\,P(\lambda_4)}\left\{1+A_1(\theta)H_1(\lambda_2,\lambda_3,\lambda_4)+A_2(\theta)H_2(\lambda_2,\lambda_3,\lambda_4)+\dots\right\},$$

a result first obtained on p. 175 of this volume, where a definition of $A_r(\theta)$ is also given.

By putting $\lambda_4=0$, we, of course, return to the formulae in the last section.

4. If in the formula § 2 (1), we put $\lambda_2 = \mu$, $\lambda_3 = \nu$, $\lambda = e^{\theta i}$, $\lambda_1 = e^{-\theta i}$, we get

$$\frac{(\mu\nu)}{P(\mu)\,P(\nu)} = 1 + H_1(\mu,\nu)\,A_1(\theta) + A_2(\mu,\nu)\,A_2(\theta) + \ldots,$$

where $$1 + \Sigma k^r H_r(\mu,\nu) \equiv \frac{1}{(k\mu)(k\nu)}.$$

This leads to an equation connecting the product of any two of the A's with a linear function of the A's.

For we have, since

$$\frac{1}{P(\mu)} = 1 + \frac{\mu}{1-q} A_1(\theta) + \ldots,$$

$$\left\{1 + \Sigma \frac{\mu^r}{(1-q^r)!} A_r(\theta) + \ldots\right\} \left\{1 + \Sigma \frac{\nu^r}{(1-q^r)!} A_r(\theta)\right\}$$

$$= \left\{1 + \Sigma \frac{\mu^r \nu^r}{(1-q^r)!}\right\} \left\{1 + \Sigma H_r(\mu,\nu)\,A_r(\theta)\right\}.$$

Equating coefficients of $\mu^r\nu^s$, where, say, $s > r$, we see that

$$\begin{aligned}\frac{A_r(\theta)\,A_s(\theta)}{(1-q^r)!\,(1-q^s)!} = {} & A_{r+s}(\theta)\,\{\text{coeff. of } \mu^r\nu^s \text{ in } H_{r+s}(\mu,\nu)\} \\ & + \frac{1}{1-q} A_{r+s-2}(\theta)\,\{\text{coeff. of } \mu^{r-1}\nu^{s-1} \text{ in } H_{r+s-2}(\mu,\nu)\} \\ & + \ldots \\ & + \frac{1}{(1-q^r)!} A_{s-r}(\theta)\,\{\text{coeff. of } \nu^{s-r} \text{ in } H_{s-r}(\mu,\nu)\}.\end{aligned}$$

From this we see that no absolute term occurs unless $r = s$, in which case the absolute term in $A_r(\theta)^2$ is $(1-q^r)!$ ....(1).

These results lead to a very interesting formula expressing the quotient of any series $K_0 + K_1 A_1(\theta) + K_2 A_2(\theta) + \ldots$ by the product $P(\lambda)$.

Let this quotient, *i.e.* the product of the K-series with

$$1 + \Sigma \frac{\lambda^r A_r(\theta)}{(1-q^r)!},$$

be denoted by $$L_0 + L_1 A_1(\theta) + \ldots.$$

In multiplying these series, we get a series of terms containing products of the form $A_r(\theta)\,A_s(\theta)$.

By (1), however, we easily see that the absolute term

$$L_0 = K_0 + K_1\lambda + K_2\lambda^2 + \ldots \quad \text{.....................(2)},$$

which may be called the generating function of the K series, and it is obvious that the generating function in λ of the KA series for $1/P(\lambda_1)\,P(\lambda_2)\,\ldots\,P(\lambda_r)$ is symmetrical in $\lambda, \lambda_1, \lambda_2, \ldots \lambda_r$, since it is the absolute term in the expansion of $1/P(\lambda)\,P(\lambda_1)\,P(\lambda_2)\,\ldots\,P(\lambda_r)$.

The following coefficient may be expressed in a very simple symbolical form involving the operation δ.

Now, supposing the K's not to contain λ, we see that

$$\delta\left\{L_0 + L_1A_1(\theta) + \ldots\right\} = \left\{K_0 + K_1A_1(\theta) + \ldots\right\}\frac{2\cos\theta - \lambda}{P(\lambda)}$$

$$= (2\cos\theta - \lambda)\left\{L_0 + L_1A_1(\theta) + \ldots\right\}.^*$$

But $$A_{r+1}(\theta) + (1-q^r)A_{r-1}(\theta) = 2\cos\theta\,.\,A_r(\theta).$$

Equating coefficients of A_r, we get

$$(1-q^{r+1})L_{r+1} = \lambda L_r - L_{r-1} + \delta L_r \quad \text{..................(3)}.$$

Thus $$(1-q)\,L_1 = (\lambda+\delta)L_0,$$

$$\begin{aligned}(1-q)(1-q^2)\,L_2 &= \lambda L_1 - (1-q)\,L_0 + (1-q)\,\delta L_1 \\ &= (L_0 - q\eta L_0) + \delta^2 L_0 \\ &= (\lambda^2+\lambda\delta)\,L_0 - (1-q)\,L_0 + L_0 - q\eta L_0 + \delta^2 L_0 \\ &= (\lambda^2+\lambda\delta)\,L_0 + qL_0 - q\,(L_0 - \lambda\delta L_0) + \delta^2 L_0 \\ &= \lambda^2 L_0 + (1+q)\,\lambda\delta L_0 + \delta^2 L_0.\end{aligned}$$

That is, L_1, L_2 are easily seen to be the $H_1(\lambda, \delta)\,L_0$ and $H_2(\lambda, \delta)\,L_0$, where H has the meaning assigned to it at the beginning of this paper. It must be noticed, however, that λ and δ are not inter-

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* For $$1/P(\lambda) = 1 + \Sigma\frac{\lambda^r A_r(\theta)}{(1-q^r)\,!};$$

therefore $$1/P(\lambda q) = (1 - 2\lambda\cos\theta + \lambda^2)/P(\lambda) = 1 + \Sigma\frac{\lambda^r q^r A_r(\theta)}{(1-q^r)\,!}.$$

Equating coefficient of λ^{r+1}, we get the relation required.

changeable, since δ operates on λ, but that λ must precede δ in all the terms.

We may now establish the general term L_r by induction. Assume

$$L_r = H_r(\lambda, \delta)\, L_0, \text{ for all values of } r \text{ up to } r \quad \text{......(4).}$$

Then $\quad (1-q^{r+1})\, L_{r+1} = (\lambda+\delta)\, H_r(\lambda, \delta)\, L_0 - H_{r-1}(\lambda, \delta)\, L_0.$

The coefficient of $\lambda^m\delta^n$ in $H_r(\lambda, \delta)$ is

$$\frac{1}{(1-q^m)!\,(1-q^n)!},$$

where $$m+n=r.$$

Now $$\delta\lambda^m\delta^n f(\lambda) = \lambda^{m-1}\delta^n f(\lambda) - \lambda^{m-1} q^m \eta\delta^n f(\lambda)$$
$$= \lambda^{m-1}\delta^n f(\lambda) - q^m \left\{\lambda^{m-1}\delta^n f(\lambda) - \lambda^m\delta^{n+1} f(\lambda)\right\}$$
$$= (1-q^m)\,\lambda^{m-1}\delta^n f(\lambda) + q^m\lambda^m\delta^{n+1} f(\lambda).$$

The first of these terms obviously cancels with the terms containing $\lambda^{m-1}\delta^n$ in $H_{r-1}(\lambda, \delta)$, so that we get

$$(1-q^{r+1})\, L_{r+1} = \Sigma\left\{\frac{q^m}{(1-q^m)!\,(1-q^n)!} + \frac{1}{(1-q^{m-1})!\,(1-q^{n+1})!}\right\}\lambda^m\delta^{n+1}L_0$$
$$= (1-q^{r+1})\,\Sigma\,\frac{\lambda^m\delta^{n+1}L_0}{(1-q^m)!\,(1-q^{n+1})!}$$
$$= (1-q^{r+1})\, H_{r+1}(\lambda, \delta)\, L_0.$$

But we have established the truth of (4) for $r=1$, $r=2$. It is therefore true universally.

We may conveniently state the result thus:—

$$\left\{K_0 + K_1A_1(\theta) + K_2A_2(\theta) + \ldots\right\} \div (1-2\lambda\cos\theta+\lambda^2)(1-2\lambda q\cos\theta+\lambda^2q^2)\ldots$$
$$= \frac{(\lambda\delta)}{P(\lambda)\,P(\delta)}\left\{K_0+K_1\lambda+K_2\lambda^2+\ldots\right\}$$
$$= \left\{1 + H_1(\lambda, \delta)\, A_1(\theta) + H_2(\lambda, \delta)\, A_2(\theta) + \ldots\right\}(K_0+K_1\lambda+\ldots) \ldots (5).$$

The generating function of

$$\left\{K_0 + K_1A_1(\theta) + \ldots\right\} / P(\lambda),$$

expressed in powers of k, is found by writing k^r for $A_r(\theta)$ in the right-hand side of (5).

This is $\{1+kH_1(\lambda,\delta)+k^2H_2(\lambda,\delta)+\dots\}(K_0+K_1\lambda+\dots)$

$$=\frac{1}{(k\lambda)(k\delta)}(K_0+K_1\lambda+\dots) \quad\dots\dots\dots\dots(6).$$

If, then, the generating function of a series in $A(\theta)$ be known, we have a convenient symbol for the generating function of the series divided by $P(\lambda)$.

5. Now we have seen that $\dfrac{(\lambda_2\lambda_3)}{P(\lambda_2)P(\lambda_3)}$ has a generating function (say in ascending powers of λ_1),

$$1+H_1(\lambda_2,\lambda_3)\lambda_1+H_2(\lambda_2,\lambda_3)\lambda_1^2+\dots=\frac{1}{(\lambda_1\lambda_2)(\lambda_1\lambda_3)}.$$

Hence the generating function in λ of $1/P(\lambda_1)P(\lambda_2)P(\lambda_3)$

$$=\frac{1}{(\lambda\lambda_1)}\cdot\frac{1}{(\lambda\delta_1)}\cdot\frac{1}{(\lambda_1\lambda_2)(\lambda_1\lambda_3)(\lambda_2\lambda_3)},$$

which, by § 2, reduces to

$$(\lambda\lambda_1\lambda_2\lambda_3)/(\lambda\lambda_1)(\lambda\lambda_2)(\lambda\lambda_3)(\lambda_1\lambda_2)(\lambda_1\lambda_3)(\lambda_2\lambda_3),$$

a symmetric function in $\lambda, \lambda_1, \lambda_2, \lambda_3$, as, by § 4 (2), we should expect to have, since the generating function in λ is the term independent of the A's in $1/P(\lambda)P(\lambda_1)P(\lambda_2)P(\lambda_3)$.

This result may be stated thus :—

$$\frac{(\lambda_1\lambda_2)(\lambda_1\lambda_3)(\lambda_2\lambda_3)}{P(\lambda_1)P(\lambda_2)P(\lambda_3)}=1+\Sigma H_r(\lambda_1\lambda_2\lambda_3/\lambda_1;\lambda_2,\lambda_3)A_r(\theta).$$

6. Similarly, we may find simple expressions in series for the generating functions in λ of $1/P(\lambda_1)P(\lambda_2)P(\lambda_3)P(\lambda_4)$ and of

$$1/P(\lambda_1)P(\lambda_2)P(\lambda_3)P(\lambda_4)P(\lambda_5).$$

Calling these functions $\phi_4(\lambda)$ and $\phi_5(\lambda)$ respectively, we get, by § 4 (6),

$$\phi_4(\lambda)(\lambda_2\lambda_3)(\lambda_2\lambda_4)(\lambda_3\lambda_4)=\frac{1}{(\lambda\lambda_1)}\cdot\frac{1}{(\lambda\delta_1)}\cdot\frac{(\lambda_1\lambda_2\lambda_3\lambda_4)}{(\lambda_1\lambda_2)(\lambda_1\lambda_3)(\lambda_1\lambda_4)}$$

$$=\frac{1}{(\lambda\lambda_1)}\cdot\frac{1}{(\lambda\delta_1)}\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_2.\,\lambda_3,\lambda_4)\lambda_1^r\} \quad \text{[see § 1 (2)]}$$

$$=\frac{1}{(\lambda\lambda_1)}\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_2,\lambda_3,\lambda_4)h_r(\lambda,\lambda_1)\}, \text{ by Lemma I.}$$

We may, however, also write these expressions in the form

$$\frac{1}{(\lambda\lambda_1)}\cdot\frac{1}{(\lambda_1\lambda_2)}\cdot\frac{1}{(\lambda\lambda_2\eta_1)}\cdot\frac{1}{(\lambda\delta_1)}\cdot\frac{(\lambda_1\lambda_2\lambda_3\lambda_4)}{(\lambda_1\lambda_3)(\lambda_1\lambda_4)} \quad \text{(by Lemma IV.)}$$

$$=\frac{1}{(\lambda\lambda_1)}\cdot\frac{1}{(\lambda_1\lambda_2)}\cdot\frac{1}{(\lambda\lambda_2\eta_1)}\left\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_3,\lambda_4)\,h_r(\lambda,\lambda_1)\right\}.$$

Now, the effect of operating with $\frac{1}{(\lambda\lambda_2\eta_1)}$ on $h_r(\lambda,\lambda_1)$ is easily obtained by consideration of the coefficient of x^r in

$$\frac{1}{(\lambda\lambda_2\eta_1)}\cdot\frac{1}{(x\lambda)(x\lambda_1)}=\frac{(x\lambda\lambda_1\lambda_2)}{(x\lambda)(x\lambda_1)(\lambda\lambda_2)},$$

which is

$$\frac{1}{(\lambda\lambda_2)}\,h_r(\lambda\lambda_1\lambda_2/\lambda,\lambda_1).$$

Hence

$$\phi_4(\lambda)=\frac{1}{(\lambda\lambda_1)(\lambda_1\lambda_2)(\lambda_2\lambda)\,.\,(\lambda_2\lambda_3)(\lambda_3\lambda_4)(\lambda_4\lambda_2)}$$
$$\times\left\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_3,\lambda_4)\,h_r(\lambda\lambda_1\lambda_2/\lambda,\lambda_1)\right\}.$$

By 4, § 2, we have seen that $\phi_4(\lambda)$ is symmetrical in $\lambda, \lambda_1, \lambda_2, \lambda_3, \lambda_4$, so that, by an interchange of suffixes, we get a rather more convenient form,

$$\frac{1}{(\lambda\lambda_1)(\lambda\lambda_2)(\lambda\lambda_3)(\lambda\lambda_4)(\lambda_1\lambda_2)(\lambda_3\lambda_4)}\left\{1+\Sigma H_r(\lambda\lambda_1\lambda_2/\lambda_1,\lambda_2)\,h_r(\lambda\lambda_3\lambda_4/\lambda_3,\lambda_4)\right\},$$

in which λ is brought more into prominence.

Equating together these unsymmetrical expressions for the symmetrical ϕ_4, we get a series of identities connecting series built up of pairs of expressions such as $H_r(\lambda\lambda_1\lambda_2/\lambda_1,\lambda_2)$.

Again,

$$\phi_5(\lambda)(\lambda_2\lambda_3)(\lambda_3\lambda_4)(\lambda_4\lambda_2)$$

$$=\frac{1}{(\lambda_5\lambda_1)}\cdot\frac{1}{(\lambda_5\delta_1)}\cdot\frac{1}{(\lambda\lambda_1)}\left\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_2,\lambda_3,\lambda_4)\,h_r(\lambda,\lambda_1)\right\}$$

$$=\frac{1}{(\lambda_5\lambda_1)(\lambda\lambda_1)}\cdot\frac{1}{(\lambda\lambda_5\eta_1)}\cdot\frac{1}{(\lambda_5\delta_1)}\left\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_2,\lambda_3,\lambda_4)\,h_r(\lambda,\lambda_1)\right\}$$
(by Lemma IV.)

$$=\frac{1}{(\lambda_5\lambda_1)(\lambda\lambda_1)}\cdot\frac{1}{(\lambda\lambda_5\eta_1)}\left\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_2,\lambda_3,\lambda_4)\,h_r(\lambda,\lambda_1,\lambda_5)\right\}$$

$$=\frac{1}{(\lambda_5\lambda_1)(\lambda\lambda_1)(\lambda\lambda_5)}\left\{1+\Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_2,\lambda_3,\lambda_4)\,h_r(\lambda\lambda_1\lambda_5/\lambda,\lambda_1,\lambda_5)\right\};$$

by the same principle as in the previous section,

$$\phi_5(\lambda) = \frac{1}{(\lambda_2\lambda_3)(\lambda_3\lambda_4)(\lambda_4\lambda_2).(\lambda\lambda_1)(\lambda_1\lambda_5)(\lambda_5\lambda)}$$

$$\times \left\{1 + \Sigma H_r(\lambda_2\lambda_3\lambda_4/\lambda_2, \lambda_3, \lambda_4)\, h_r(\lambda\lambda_1\lambda_5/\lambda, \lambda_1, \lambda_5)\right\}.$$

Here again we get an only partially symmetric series for the entirely symmetrical function ϕ_5, so that, by interchanging suffixes, we get a set of transformation-formulae connecting such series in general. These transformations bear a kind of analogy to Heine's.

7. Difference equations for generating functions.

It has been shown that the generating function of $1/P(\lambda_1)P(\lambda_2)$ and $1/P(\lambda_1)P(\lambda_2)P(\lambda_3)$ may be conveniently expressed as infinite products. It is not, however, possible so to express the generating functions of $1/P(\lambda_1)P(\lambda_2)P(\lambda_3)P(\lambda_4)$, &c., but by a simple process we may obtain successively functional equations which such generating functions satisfy.

The generating function, $\phi_3(\lambda)$ say, in λ, for $1/P(\lambda_1)P(\lambda_2)P(\lambda_3)$, found above, in § 5, evidently satisfies the functional equation

$$\phi_3(\lambda)(1-\lambda\lambda_1)(1-\lambda\lambda_2)(1-\lambda\lambda_3) = \phi_3(\lambda q)(1-\lambda\lambda_1\lambda_2\lambda_3) \quad \text{......(1)}.$$

Now, $\phi_3(\lambda q)$ can be made to depend on $\delta\phi_3(\lambda)$, while the latter may be made to depend on the coefficient of k in the k-generating function for $1/P(\lambda)P(\lambda_1)P(\lambda_2)P(\lambda_3)$, by § 4.

If we call this latter generating function

$$K_0 + \frac{kK_1}{1-q} + \ldots,$$

the relation (1) can be transformed into an equation connecting K_0 and K_1, which will be symmetrical in $\lambda, \lambda_1, \lambda_2, \lambda_3$.

We get from (1),

$$\phi_3(\lambda)\left\{\lambda_1+\lambda_2+\lambda_3-\lambda(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)+\lambda^2\lambda_1\lambda_2\lambda_3-\lambda_1\lambda_2\lambda_3\right\}$$

$$= (1-\lambda\lambda_1\lambda_2\lambda_3)\,\delta\phi_3(\lambda).$$

Now, by § 4 (3), $\phi_3(\lambda) = K_0$, and $(\lambda+\delta)\,\phi_3(\lambda) = K_1$;

whence $$(p_1-p_3)K_0 = (1-p_4)K_1 \quad \text{........................(2)},$$

where p_1, p_3, p_4 are the coefficients in the equation whose roots are $\lambda, \lambda_1, \lambda_2, \lambda_3$.

From (2) we shall easily derive a connection between the first three coefficients of the generating function for

$$1/P(\lambda)\,P(\lambda_1)\,P(\lambda_2)\,P(\lambda_3)\,P(\lambda_4),$$

which may be reduced, by § 4 (5), to a functional equation in $\phi_4(\lambda)$, $\delta\phi_4\lambda$, $\delta^2\phi_4(\lambda)$.

Now, by § 4 (6), we have seen that, if

$$K_0+\frac{K_1}{1-q}\lambda+\dots$$

be the λ-generating function of $1/P(\lambda_1)\,P(\lambda_2)\,P(\lambda_3)$, then the k-generating function of $1/P(\lambda)\,P(\lambda_1)\,P(\lambda_2)\,P(\lambda_3)$ is

$$\frac{1}{(k\lambda)(k\delta)}\left\{K_0+\frac{K_1}{1-q}\lambda+\dots\right\}=L_0+\frac{L_1}{1-q}k+\dots \text{ say,}$$

and that $$L_0=K_0+\frac{K_1}{1-q}\lambda+\dots=\frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}K_0\dots\dots\dots\dots\dots\dots(3).$$

Now

$$\frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}K_1=\frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}(\lambda_1+\delta_1)\,K_0, \text{ by § 4 (3),}$$

$$=\frac{\lambda_1}{(\lambda\lambda_1)(\lambda q\delta_1)}K_0+\frac{\lambda}{(\lambda\lambda_1)(\lambda q\delta_1)}K_0+\frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}\delta_1K_0$$

(by Lemma III.)

$$=\frac{\lambda_1}{(\lambda\lambda_1)(\lambda q\delta_1)}K_0+\lambda L_0-\frac{1}{\lambda}\cdot\frac{1-\lambda\delta_1}{(\lambda\lambda_1)(\lambda\delta_1)}K_0+\frac{1}{\lambda\,(\lambda\lambda_1)(\lambda\delta_1)}K_0$$

$$=\frac{\lambda_1}{(\lambda\lambda_1)(\lambda q\delta_1)}K_0-\frac{1}{\lambda\,(\lambda\lambda_1)(\lambda q\delta_1)}K_0+\frac{1}{\lambda}\lambda L_0+\frac{1}{\lambda}L_0$$

$$=-\frac{1}{\lambda\,(\lambda q\lambda_1)(\lambda q\delta_1)}K_0+\lambda L_0+\frac{1}{\lambda}L_0$$

$$=(\lambda+\delta)\,L_0=L_1.$$

Similarly, if we assume

$$\frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}K_r=L_r\dots\dots\dots\dots\dots\dots\dots\dots(4),$$

we can prove inductively by precisely similar steps; and, remembering that, by § 4 (3),

$$K_{r+1}=(\lambda_1+\delta_1)\,K_r-(1-q^r)\,K_{r-1},$$

since we have replaced K_r by $K_r/(1-q^r)!$, and that a similar equation in the L's holds good, we get

$$\frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}K_{r+1}=L_{r+1}.$$

Hence (4) holds good for all values of r.

$$\text{Again, } \frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}\lambda_1 K_r=\frac{\lambda_1}{(\lambda\lambda_1)(\lambda q\delta_1)}K_r+\lambda L_r \quad \text{(by Lemma III.)}$$

$$=\lambda L_r+\frac{\lambda_1}{1-\lambda\lambda_1}\eta L_r$$

$$=\frac{(\lambda+\lambda_1)L_r-\lambda\lambda_1(\lambda+\delta)L_r}{1-\lambda\lambda_1}$$

$$=\frac{(\lambda+\lambda_1)L_r-\lambda\lambda_1\{L_{r+1}+(1-q^r)L_{r-1}\}}{1-\lambda\lambda_1}\quad\text{...(5).}$$

We are now in a position to derive an equation connecting L_0, L_1, L_2. Operating with $\frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}$ on the equation

$$(p_1-p_3)K_0-(1-p_4)K_1=0,$$

where now p_1, p_3, p_4, K_0, K_1 refer to λ_1, λ_2, λ_3, λ_4 instead of to λ, λ_1, λ_2, λ_3, we see, by the help of (4) and (5) and § 4 (6), that

$$\begin{aligned}(1-\lambda_2\lambda_3-\lambda_2\lambda_4-\lambda_3\lambda_4)&\{(\lambda+\lambda_1)L_0-\lambda\lambda_1L_1\}\\&+L_0(1-\lambda\lambda_1)(\lambda_2+\lambda_3+\lambda_4-\lambda_2\lambda_3\lambda_4)-(1-\lambda\lambda_1)L_1\\&+\lambda_2\lambda_3\lambda_4\{(\lambda+\lambda_1)L_1-\lambda_1\lambda_2L_2-\lambda_1\lambda_2(1-q)L_0\}=0,\end{aligned}$$

which easily reduces to

$$(p_1-p_3+p_5q)L_0-(1-p_4)L_1-p_5L_2=0\quad\text{..............(6),}$$

where now p_1, p_3, p_4, p_5 are coefficients in the equation whose roots are λ, λ_1, λ_2, λ_3, λ_4, so that $p_1=\lambda+\lambda_1+\lambda_2+\lambda_3+\lambda_4$, &c.

Replacing L_0 by $\phi_4(\lambda)$, L_1 by $(\lambda+\delta)\,\phi_4(\lambda)$, &c., by § 4 (5), we get

$$\begin{aligned}(1-\lambda\lambda_1)(1-\lambda\lambda_2)(1-\lambda\lambda_3)(1-\lambda\lambda_4)\,\phi_4(\lambda)&\\-\{1+\lambda_1\lambda_2\lambda_3\lambda_4q^{-1}-\lambda\,(\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4)&\\+\lambda^2(1+q)\,\lambda_1\lambda_2\lambda_3\lambda_4\}\,\phi_4(\lambda q)&\\+\lambda_1\lambda_2\lambda_3\lambda_4q^{-1}\phi_4(\lambda q^2)=0\quad\text{..(7),}&\end{aligned}$$

a functional equation determining the generating function of $1/P(\lambda_1)P(\lambda_2)P(\lambda_3)P(\lambda_4)$, and identical with the expression given in § 6.

Again, if we increase by unity all the suffixes of the λ's involved in (6), and operate on the equation with $\frac{1}{(\lambda\lambda_1)(\lambda\delta_1)}$, we shall get a relation connecting M_0, M_1, M_2, M_3, where

$$M_0+\frac{M_1}{1-q}k+\ldots$$

is the generating function of

$$1/P(\lambda)P(\lambda_1)P(\lambda_2)P(\lambda_3)P(\lambda_4)P(\lambda_5).$$

This is found to be

$$(p_1-p_3+p_5q)M_0-\{1-p_4+q(1+q)p_6\}M_1-M_2p_5+M_3p_6=0\ldots(8),$$

where $\quad p_1=\lambda+\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5$, &c.

This can be reduced to a functional equation connecting $\phi_5(\lambda)$, $\phi_5(\lambda q)$, $\phi_5(\lambda q^2)$, and $\phi_5(\lambda q^3)$.

It will be easily seen, moreover, that, since the p's only contain λ_1 to the first degree, we can always derive new equations of the type of (6) and (8) by help of (4) and (5), and will obtain linear relations between a finite number of coefficients of the generating functions, there being $r-1$ coefficients in the case of $1/P(\lambda)P(\lambda_1)\ldots P(\lambda_r)$. These will be multiplied by linear functions of the symmetrical functions of $\lambda, \lambda_1, \ldots \lambda_r$.

[8. By employing an operative symbol δ defined by the equation

$$\delta f(\lambda)=\frac{f(\lambda/q)-f(\lambda)}{\lambda},$$

so that $\quad \delta\lambda^r q^{\frac{1}{2}r(r+1)}=\lambda^{r-1}q^{\frac{1}{2}r(r-1)}(1-q^r),$

and

$$(-\lambda q\delta_1)\frac{\lambda_1^r q^{\frac{1}{2}r(r+1)}}{(1-q^r)!}\equiv\left\{1+\frac{q}{1-q}\lambda\delta_1+\frac{q^3}{(1-q)(1-q^2)}\lambda^2\delta_1^2+\ldots\right\}\frac{\lambda_1^r q^{\frac{1}{2}r(r+1)}}{(1-q^r)!}$$

$$=\text{coef. of } x^r \text{ in } (-\lambda qx)(-\lambda_1 qx),$$

we see that $\quad (-\lambda q\delta_1)(-\lambda_1\lambda_2 q)=(-\lambda\lambda_2 q)(-\lambda_1\lambda_2 q)$ ....(1).

Moreover, corresponding to Lemmata II., III, and IV., in § 1, we get

$$\frac{1}{(\lambda\delta_1)} = \frac{1}{\left(-\frac{\lambda}{\lambda_1}\right)}\left(-\frac{\lambda}{\lambda_1\eta_1}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$(-\lambda q\delta_1)(\lambda_1-\lambda)\,\psi(\lambda_1) = \lambda_1(-\lambda\delta_1)\,\psi(\lambda_1) \quad\ldots\ldots\ldots\ldots\ldots(3),$$

and $(-\lambda q\delta_1)(-\lambda_1\mu q)\,\psi(\lambda_1) = (-\lambda_1\mu q)\left(-\frac{\lambda\mu q}{\eta_1}\right)(-\lambda q\delta_1)\,\psi(\lambda_1) \ \ldots \ (4).$

By the help of these results we deduce that, if $B_r(\theta)$ be defined by the equation

$$P(-\lambda q) \equiv 1 + \frac{B_1(\theta)}{1-q}\lambda + \frac{B_2(\theta)}{(1-q)(1-q^2)}\lambda^2 + \ldots,$$

then $P(-xq,\, \theta+\phi)\, P(-xq,\, \theta-\phi)/(x^2q)$

$$= 1 + \frac{B_1(\theta)\, B_1(\phi)\, q^{-1}}{1-q} + \frac{B_2(\theta)\, B_2(\phi)}{(1-q)(1-q^2)}\, q^{-3} + \ldots,$$

and that $\dfrac{P(-\lambda_1 q)\, P(-\lambda_2 q)\, P(-\lambda_3 q)}{(\lambda_2\lambda_3 q)(\lambda_3\lambda_1 q)(\lambda_1\lambda_2 q)}$

$$= 1 + \Sigma H_r(\lambda_1 q,\, \lambda_2 q,\, \lambda_3 q/\lambda_1\lambda_2\lambda_3 q)\, q^{-\frac{1}{2}r(r+1)}\, B_r(\theta),$$

corresponding to § 5.]

Note on some Properties of Gauche Cubics. *By* T. R. Lee.

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1. If a system of quadrics pass through seven points taken arbitrarily in space, they have also one other point common (Salmon's *Geometry of Three Dimensions*, p. 91, first edition, pp. 97, 98, third edition).

This eighth point might be called the eighth point homologous to the seven given points, while the system of eight points might be called eight homologous points.

It is the object of this note to prove, by purely geometrical methods,

(1) the existence of the eighth homologous point, in a certain class of cases,

(2) certain properties belonging to such a system of eight points, and

(3) two properties of gauche cubics—one of them analogous to Desargues' theorem, that "a chord of a conic is cut in involution by the opposite sides of an inscribed quadrilateral"; the other property being the converse of the first, and affording us a test by which we may determine whether a given line is a chord of a given cubic or not.

It will be found that the theory of eight homologous points is closely connected with the theory of gauche cubics.

2. As the demonstrations in this paper are to be purely geometrical, we are not at liberty to assume the theorem stated above relating to a system of quadrics passing through seven points. We are obliged, therefore, to put the definition of an eighth homologous point into a hypothetical form. We shall use the following definition:—

Def. 1.—If three quadrics pass through seven given points in space, and if they have one, and only one, other point common, this eighth point is called the eighth homologous point with respect to the seven given points and the three given surfaces.

It is convenient, also, to use the following definitions:—

Def. 2.—When a system of six points in space is given, if we draw a plane through any three of them, and then draw a second plane through the three remaining points, these two planes are called a pair of opposite planes of the system.

Thus, if the points be 1, 2, 3, 4, 5, 6; then 235 and 461 are opposite planes, and 345, 612 are opposite planes, &c.

Def. 3.—Given a system of points, the line joining any two of them is called a line of the system.

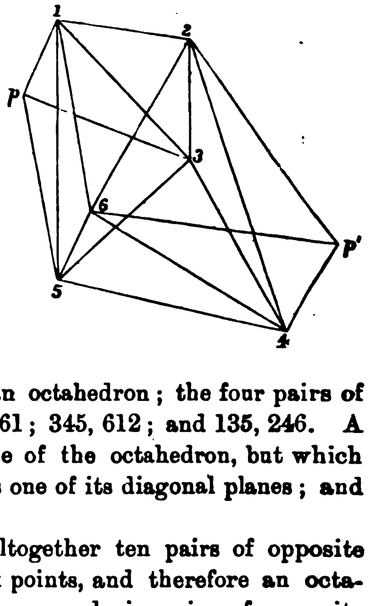
Def. 4.—The plane determined by two successive sides of a gauche polygon is called a face of the polygon.

The polygon, therefore, has as many faces as it has sides or corners.

[redacted]

Def. 5.—A diagonal plane of a polyhedron is any plane determined by three of the corners which are not in the same face.

3. Thus, in a gauche hexagon 123456, the faces are the planes 123, 234, 345, 456, 561, 612. The opposite faces are 123, 456; 234, 561; 345, 612.



There are two triplets of alternate faces: 123, 345, 561, which intersect in p; and 234, 456, 612, which intersect in p'. The whole figure forms a hexahedron, whose opposite faces are the same as the opposite faces of the hexagon. The points p, p' might be called the poles of the hexagon.

If we take the pair of opposite planes 135, 246 (Def. 2), determined by the two sets of alternate corners of the hexagon, we have a system of eight planes forming an octahedron; the four pairs of opposite faces being 123, 456; 234, 561; 345, 612; and 135, 246. A plane such as 124, which is not a face of the octahedron, but which passes through three of its corners, is one of its diagonal planes; and the opposite diagonal plane is 356.

It is easy to see that there are altogether ten pairs of opposite planes determined by a system of six points, and therefore an octahedron has four pairs of opposite faces, and six pairs of opposite diagonal planes.

4. Whenever the term "hyperboloid" is used in this paper, what is meant is "hyperboloid of one sheet." This will include a "hyperbolic paraboloid," which is nothing but a hyperboloid of one sheet having one generator at infinity.

A hyperboloid is the locus of lines which intersect three fixed lines. From this definition is easily deduced the anharmonic property that "A hyperboloid is generated by the intersections of two homographic systems of planes, which turn round two fixed lines" (Chasles, *Géometrie Supérieure*, Chap. xx., No. 411, 1re édition; No. 420, 2me édition).

If the two fixed lines intersect each other, the hyperboloid becomes *a cone;* and a cone is, therefore, generated by the intersections of

two homographic systems of planes, which turn round two fixed intersecting lines. This property of a cone of the second degree may be regarded as a consequence of the anharmonic property of a conic, namely, that "A conic is the locus of the intersection of two homographic systems of lines, which turn in one plane round two fixed points." Now the anharmonic property of a conic can be expressed in this way: "The anharmonic ratio of the lines joining a point on a conic to four fixed points on the curve, is constant." And, in the same way, the anharmonic property of a cone of the second degree may be expressed thus: "The anharmonic ratios of the planes determined by a generator of a cone of the second degree and four fixed generators of the surface, is constant."

5. If 1, 2, 3, 4 are four points in a line, then (1234) denotes their anharmonic ratio. If 1, 2, 3, 4 are any four points in space, and L is any line, then L (1234) denotes the anharmonic ratio of the four planes $L1$, $L2$, $L3$, $L4$.

§ I.

6. *Lemma A.*—The intersection of two quadrics is a quartic curve.

A quadric is a surface which is met by any straight line in two points; and, consequently, its intersection with a plane is a conic. Draw any plane intersecting the two given quadrics in two conics. These two conics intersect in four points (Chasles, *Traité des Sections Coniques*, Chap. XIII., Nos. 328, 335); these four points are common to the two quadrics, and, therefore, lie on their curve of intersection. Therefore every plane meets the curve of intersection of two quadrics in four points, *i.e.*, the curve of intersection is a quartic.

7. The intersection of two quadrics is not in all cases a proper curve of the fourth degree. It may break up into four lines. Thus two hyperboloids may pass through a gauche quadrilateral. Again, if the two quadrics are ruled surfaces, and have a common generator, their intersection consists of this generator and a gauche cubic.

It is worth noticing here that every gauche cubic may be considered as the partial intersection of two quadrics having a common generator. For the cone containing a gauche cubic and having its vertex on the curve is easily seen to be a quadric, and the intersection of two such cones is the cubic itself and the common generator joining their vertices.*

* Hence we infer that only one gauche cubic can be described through six given points; or, six points determine a gauche cubic (Salmon, *Geometry of Three Dimensions*, No. 302, first edition; No. 365, third edition).

[redacted]

Hence, with the help of the anharmonic properties of cones and hyperboloids, it is easy to show that a hyperboloid can be constructed which will pass through the gauche cubic and have any two given chords of the curve as generators of the same system; from which it follows that a gauche cubic may be considered as the partial intersection of two hyperboloids, or of a hyperboloid and a cone.

8. *Lemma B.*—Through seven points in space it is, in general, possible to describe an infinite number of hyperboloids.

If the seven given points are 1, 2, 3, 4, 5, 6, 7, and if we take any two lines of the system 12, 34 as fixed axes, and draw two variable planes π, π' passing, respectively, through 12 and 34, in such a manner that the anharmonic ratio of the four planes 125, 126, 127, and π is equal to the anharmonic ratio of the four planes 345, 346, 347, and π', the locus of the intersection of π and π' will be a hyperboloid passing through the seven given points (No. 4).

Next, let L be any line drawn through one of the points 1, and let us examine if it be possible to draw a line M through 2, so that the system of planes $L3$, $L4$, $L5$, $L6$, $L7$ may be homographic with the system $M3$, $M4$, $M5$, $M6$, $M7$. Using the notation of No. 5, we are to have

$$L\,(3456) = M\,(3456),$$

and also

$$L\,(3457) = M\,(3457).$$

The first equality shows, since $L\,(3456)$ is given, that $M\,(3456)$ is given, and, therefore, M is a generator of a known cone of the second degree, having also 23, 24, 25, 26 for generators (No. 4). The second equality shows that M is a generator of a known cone having 23, 24, 25, 27 for generators. Now two cones having a common vertex have four generators common, because their sections by any plane have four points common; and these two cones have three generators common; therefore M is their fourth common generator. And it is easy to determine (by a construction with the ruler alone) the fourth point of intersection of the two conics which are the sections of these two cones by the plane 345. We can thus determine M without difficulty; and it is evident (by No. 4) that the hyperboloid having L and M as generators, and passing through 3, 4, 5, will also pass through 6 and 7. In what follows we denote by $(ab, a'b')(c, c', x)$ the hyperboloid having the lines ab, $a'b'$ for generators of one system, and passing through the points c, c', x.

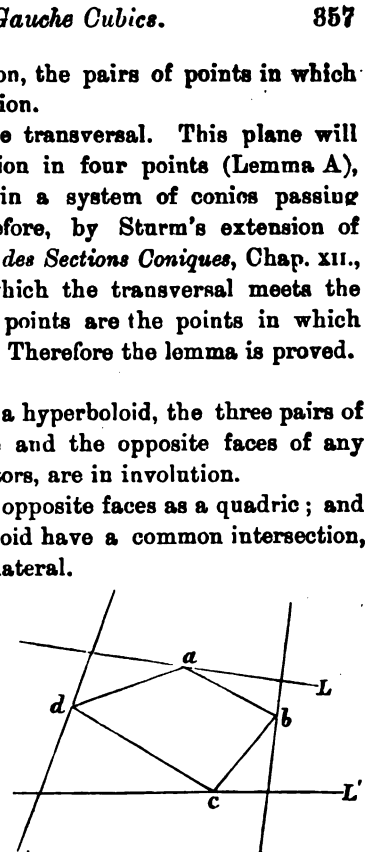
9. *Lemma C.*—If a transversal intersect a system of quadrics

having a common curve of intersection, the pairs of points in which it meets those surfaces are in involution.

For, draw any plane through the transversal. This plane will meet the common curve of intersection in four points (Lemma A), and it will intersect the quadrics in a system of conics passing through those four points. Therefore, by Sturm's extension of Desargues' theorem (Chasles, *Traité des Sections Coniques*, Chap. XII., No. 302), the pairs of points in which the transversal meets the conics are in involution. But these points are the points in which the transversal meets the quadrics. Therefore the lemma is proved.

Cor. 1.—If a transversal intersect a hyperboloid, the three pairs of points, in which it cuts the surface and the opposite faces of any quadrilateral formed by four generators, are in involution.

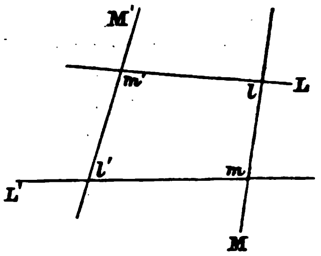
For we may consider each pair of opposite faces as a quadric; and these two quadrics and the hyperboloid have a common intersection, namely, the four sides of the quadrilateral.



We may also prove this directly, as we proved the lemma, by drawing a plane π through the transversal. Thus, if L, M, L', M' be the four generators forming the quadrilateral, and if π meet those generators in a, b, c, d respectively, and the hyperboloid in a conic C which passes through a, b, c, d; by Desargues' theorem, the three pairs of points, in which the transversal meets the conic C and the opposite sides of the plane quadrilateral $abcd$, are in involution. Now the points in which the transversal meets C are the points in which it meets the hyperboloid; and the points in which it meets the opposite sides of the plane quadrilateral $abcd$ are the points in which it meets the opposite faces of the gauche quadrilateral $LML'M'$; which proves the theorem.

We may also deduce this corollary directly from the anharmonic property of a hyperboloid (No. 4), just as Chasles deduces Desargues' theorem from the anharmonic property of conics (*Traité des Sections Coniques*, Chap. II., No. 20). If we conceive two planes turning round L and L', in such a manner as to always intersect in a second generator of the surface, they will form on the transversal two homographic systems of points, whose double points e, f (real or

imaginary) will be the points of the hyperboloid situated on the transversal. Now, if the transversal cut the two pairs of opposite faces of the quadrilateral in l, l'; m, m' respectively, l, m and m', l' are two pairs of corresponding points of the two homographic systems on the transversal. Therefore, the three pairs l, l'; m, m', and e, f are in involution (*Géométrie Supérieure*, 259, 1[re] édition, 267, 2[me] édition).

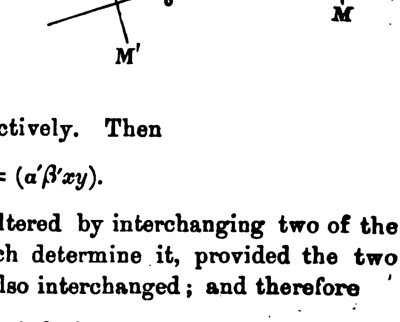


Cor. 2.—Conversely, if a transversal meet a hyperboloid in a point e, and the opposite faces of a quadrilateral formed by four generators in l, l' and m, m'; and if we take f the conjugate of e in the involution determined by ll' and mm', then f also lies on the hyperboloid.

Cor. 3.—Given a gauche hexagon $abca'b'c'$, and a transversal xy (meeting its faces, not its sides, as in a plane polygon),

if $\quad ab\,(cc'xy) = a'b'\,(cc'xy)$,

then the points x, y and the two pairs of points in which the transversal meets the opposite faces $c'ab$, $ca'b'$ and abc, $a'b'c'$, are in involution.



Let xy meet the planes abc, $a'b'c'$, abc', $a'b'c$ in β, β', α, α', respectively. Then

$$(\beta\alpha xy) = (\alpha'\beta' xy).$$

Now an anharmonic ratio is not altered by interchanging two of the four points, lines, or planes, which determine it, provided the two other points, lines, or planes are also interchanged; and therefore

$$(\beta\alpha xy) = (\alpha\beta yx).$$

Consequently $\qquad (\alpha\beta yx) = (\alpha'\beta' xy),$

and therefore the three pairs of points x, y; α, α'; β, β' are in involution.

Otherwise, thus:—Draw M, M' through c, c' respectively, to meet

ab and $a'b'$. Then (Lemma B) a hyperboloid can be described, having ab, $a'b'$ as generators, and passing through c, c', and x; and, since

$$ab\,(cc'xy) = a'b'\,(cc'xy),$$

it will also pass through y. Therefore, by Cor. 1, the pair of points x, y, and the two pairs of points in which xy meets the opposite faces of the quadrilateral formed by ab, M, $a'b'$, M', are in involution. But the two last pairs of points are the points in which xy meets the opposite faces $c'ab$, $ca'b'$ and abc, $a'b'c'$ of the hexagon. The corollary is, therefore, proved.

Cor. 4.—Conversely, if xy and the two pairs of points, in which xy is cut by the opposite faces $c'ab$, $ca'b'$ and abc, $a'b'c'$ of the hexagon $abca'b'c'$, are in involution, then

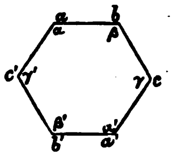
$$ab\,(cc'xy) = a'b'\,(cc'xy).$$

This is seen to be true by reversing the steps of the first proof of Cor. 3; or it may be deduced from Cor. 2, as Cor. 3 was deduced from Cor. 1.

10. *Lemma D.*—Given a gauche hexagon $abca'b'c'$, and a transversal xy meeting its opposite faces in α, α'; β, β'; γ, γ'; if

$$ab\,(cc'xy) = a'b'\,(cc'xy), \quad \text{and} \quad bc\,(aa'xy) = b'c'\,(aa'xy);$$

then $$ca'\,(bb'xy) = c'a\,(bb'xy).$$



By Lemma C, Cor. 3, $\alpha\alpha'$, $\beta\beta'$, xy are three segments in involution; and $\beta\beta'$, $\gamma\gamma'$, xy are three segments in involution; therefore $\gamma\gamma'$, $\alpha\alpha'$, xy are three segments in involution; and therefore, by Lemma C, Cor. 4,

$$ca'\,(bb'xy) = c'a\,(bb'xy).$$

Cor. 1.—Conversely, being given two of the three involutions just mentioned, the three equalities of anharmonic ratios hold good. For, from two of these involutions the third follows; and, consequently, by Lemma C, Cor. 4, the three equalities of anharmonic ratios.

Cor. 2.—Using the notation indicated at the end of No. 8, the three hyperboloids $(ab,\, a'b')\,(c,\, c',\, x)$, $(bc,\, b'c')\,(a,\, a',\, x)$, $(ca',\, c'a)\,(b,\, b',\, x)$ all pass through the seven points a, b, c, a', b', c', x; and, if y be any point on the intersection of two of them

$$(ab,\, a'b')(c,\, c',\, x) \quad \text{and} \quad (bc,\, b'c')(a,\, a',\, x),$$

we have the two equalities

$$ab\,(cc'xy) = a'b'\,(cc'xy) \quad \text{and} \quad bc\,(aa'xy) = b'c'\,(aa'xy)\,;$$

therefore, by the lemma, we have also

$$ca'\,(bb'xy) = c'a\,(bb'xy),$$

and, consequently, y lies also on the third hyperboloid $(ca', c'a)(b, b', x)$. The three hyperboloids have, then, a common intersection, which (Lemma A) is a quartic curve.

11. *Lemma E.*—If two quadrics have a common generator, their intersection consists of this generator and a cubic curve, and the common generator is a chord of the cubic (No. 7).

It is a known theorem that a cubic lying on a hyperboloid meets the generators of one system once, and those of the other system twice (Salmon, *Geometry of Three Dimensions*, No. 303, pp. 242, 243, first edition; No. 366, p. 305, second edition). The present lemma follows easily, as a limiting case, from that theorem. As, however, the lemma is important, a proof is here given in full.

Let L be a generator common to two hyperboloids H_1, H_2; L_1, L_2 generators of the same system as L in H_1, H_2 respectively. Draw any plane π through L, and let π_1, π_2 be the planes through L_1 and L_2 corresponding to π in the homographic systems which determine H_1, H_2 respectively; so that M_1, the intersection of π and π_1, is a second generator of H_1; and M_2, the intersection of π and π_2, is a second generator of H_2. Let M_1, M_2 meet L in l_1, l_2, respectively.

Then M_1 and M_2, being both in one plane π, intersect each other in a point p. Now the section of H_1 by π consists of the two lines L and M_1, and the section of H_2 by π consists of the two lines L and M_2. The point p and the points of the line L are, therefore, the only points common to H_1 and H_2 in that plane. And p does not, in general, lie on L; therefore p lies on the cubic which forms part of the intersection of H_1 and H_2. But π must meet the cubic in three points (one of those points being p), and those three points must be all common to H_1 and H_2. Hence the two other points in which π meets the cubic must lie on L. The positions of those points can be found in this way: The planes π and π_1 correspond homographically, and the same is true of the planes π and π_2 (No. 4); therefore the planes π_1 and π_2 correspond homographically, and the points l_1 and l_2 form two homographic divisions on the line L. The points l_1 and l_2 coincide at the double points of these two divisions; and when they coincide p also coincides with them. Thus the

cubic cuts L at the double points of the two homographic divisions l_1 and l_2.

12. *Lemma F.*—Given seven points in space, 1, 2, 3, 4, 5, 6, 7, the three hyperboloids (12, 34)(5, 6, 7), (34, 56)(1, 2, 7), (56, 12)(3, 4, 7) have one, and only one, other point common (see end of No. 8 for explanation of the notation).

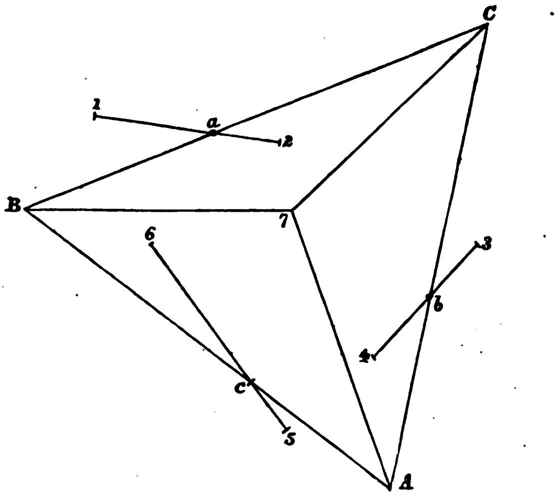
The enunciation assumes that the three hyperboloids can be constructed, that they have not a common curve of intersection, and that they do not all coincide. In general, the given points being taken arbitrarily, these conditions are fulfilled. The conditions would not be satisfied if five or more of the points lay in a plane. If five of the points are co-planar, every quadric through the seven points passes through the conic determined by the five points; and if six or seven points lie in a plane it may not be possible to describe a quadric through them. These particular cases are excluded, and also those cases in which three of the points lie in a straight line. If four of the points are co-planar, the hyperboloids may resolve into cones, or into pairs of planes. Thus, if 1, 2, 3, 4 are in a plane, the hyperboloid (12, 34)(5, 6, 7) becomes a cone, while the two other hyperboloids (34, 56)(1, 2, 7) and (56, 12)(3, 4, 7) both coincide with the two planes 1234 and 567. It does not follow, however, that three hyperboloids cannot be described through the seven points, which will also pass through one eighth homologous point (Def. 1). Thus, if 1, 3, 5, 7 are in one plane, the construction given below can be performed, and the eighth homologous point exists. We now proceed to prove the lemma.

Let $7A$ be the generator through 7 of the first system in the hyperboloid (34, 56)(1, 2, 7); and $7B$, $7C$ be generators of the first system in the hyperboloids (56, 12)(3, 4, 7) and (12, 34)(5, 6, 7), respectively.

If 8 be a point common to the three hyperboloids, and if we draw through 8 the generators of the second system in the three surfaces, these second generators will meet the first generators already drawn through 7. Let A, B, C be the points of meeting. Then $8B$, $8C$, being second generators in (12, 56)(3, 4, 7) and (12, 34)(5, 6, 7) respectively, must meet 12, which is a first generator in both hyperboloids, and thus 12 lies in the plane ($8B$, $8C$), and, consequently, BC intersects 12 in a point a. Similarly, OA intersects 34 in a point B, and AB intersects 56 in a point c. Hence a, b, c are the points where the known planes ($7B$, $7C$), ($7C$, $7A$), ($7A$, 7B) are cut by the

[redacted]

12, 34, 56; and a, b, c are points on the sides of the triangle ABC;



therefore A, B, C are the points where the plane of the known triangle abc cuts the known lines $7A$, $7B$, $7C$. It follows that 8 is the intersection of three known planes (12, BC), (34, CA), and (56, AB). This is a perfectly definite construction, and the above reasoning shows that the three hyperboloids have only one other point common besides the seven given points.

This construction was given by Th. Reye, in 1887, in the *Journal für die Mathematik*, Vol. c., p. 489; see also *Jahrbuch über die Fortschritte der Mathematik*, Baud xix., p. 637 (1887).

13. The point 8 is, by Def. 1, the eighth homologous point with respect to the seven given points and the three hyperboloids (12,34)(5, 6, 7), (34, 56)(1, 2, 7), (56, 12)(3, 4, 7). It has not been proved that the other quadrics, which pass through the seven given points, pass also through 8. The existence of the eighth homologous point has only been proved for the three hyperboloids just mentioned.

§ II.

14. *Theorem I.*—The ten pairs of points in which a chord of a cubic meets the opposite planes of a system of six points lying on the *curve, and the pair of points in which the chord meets the curve, all belong* to one involution.

Let nn' be a chord of the cubic which passes through 1, 2, 3, 4, 5, 6. Complete the hexahedron $123456pp'$ as in the figure of No. 3, *supra*. Consider the hyperboloid through the lines pp', 23, 56. The lines $p1$, $4p'$ are generators of the second system, and the four lines 23, $1p$, 56, $4p'$ are four generators which form a gauche quadrilateral. Therefore, by Lemma C, Cor. 1, the points where nn' meets the hyperboloid, the points where it meets the planes 123, 456, and the points where it meets the planes 234, 561, are three pairs of points in involution. Similarly, the points where nn' meets the hyperboloid through the lines pp', 34, 61, the points where it meets the planes 234, 561, and the points where it meets the planes 345, 612, are three pairs of points in involution.

Now, the two hyperboloids have a common generator pp'; therefore the rest of their intersection is a gauche cubic (No. 7). This cubic passes through the six points 1, 2, 3, 4, 5, 6, because they are common to both hyperboloids, and thus coincides with the given cubic, since only one cubic can be drawn through six points (footnote, No. 7). Consequently, the points n, n', which lie on the cubic (123456), also lie on both hyperboloids, and the two involutions just mentioned have a common segment nn'. They have, besides, another common segment, namely, the segment between the points where nn' cuts the opposite faces 234, 561. Having, then, two common segments, the two involutions are one and the same.

Therefore, the points n, n', where the chord meets the cubic, and the three pairs of points in which it meets the opposite faces of the hexahedron $123456pp'$, are in involution. Or, we may state the result thus: 123456 is a hexagon inscribed in a cubic; if nn' be a chord of the cubic, then the points n, n', and the three pairs of points in which nn' meets the opposite faces of the hexagon, belong to one involution.

Every hexagon that can be formed by taking 1, 2, 3, 4, 5, 6 in different orders, gives rise to an involution. Now, the nine hexagons, distinguished by Roman numerals in the table given below, have one pair of opposite faces common, namely, the pair 123, 456; therefore the involutions determined by those hexagons have one segment common. The segment nn' is also common to all those involutions. The nine involutions have, then, two common segments; and, consequently, are one and the same. Now there are ten different pairs of opposite planes in the table, and only ten pairs of opposite planes can be formed from six points. The table thus includes all the pairs of opposite planes that can be formed from the six points.

[redacted]

fore, it is proved that the points n, n', and the ten pairs of points in which the chord nn' meets the opposite planes of the system 1, 2, 3, 4, 5, 6, belong to one and the same involution.

This is the theorem, analogous to that of Desargues, which we undertook to prove. The converse of this is given below (Theorem IV., No. 18).

Hexagon.	Pairs of opposite faces, or planes, distinguished from each other by numbers in brackets.		
I. 123456	(1) 123, 456;	(2) 234, 561;	(3) 345, 612
II. 231456	(1) 231, 456;	(4) 314, 562;	(5) 145, 623
III. 312456	(1) 312, 456;	(6) 124, 563;	(7) 245, 631
IV. 123564	(1) 123, 564;	(8) 235, 641;	(6) 356, 412
V. 231564	(1) 231, 564;	(9) 315, 642;	(2) 156, 423
VI. 312564	(1) 312, 564;	(10) 125, 643;	(4) 256, 431
VII. 123645	(1) 123, 645;	(5) 235, 451;	(10) 364, 512
VIII. 231645	(1) 231, 645;	(7) 316, 452;	(8) 164, 523
IX. 312645	(1) 312, 645;	(3) 126, 453;	(9) 264, 531

15. If we refer to the construction of an octahedron in No. 3, we see that the above theorem can be stated thus:

If the corners of an octahedron rest on a cubic, the four pairs of opposite faces of the octahedron, and its six pairs of opposite diagonal planes, determine on any chord of the curve ten pairs of points belonging to the same involution; and to this involution also belong the points in which the chord meets the curve.

16. *Theorem II.*—If the points 7, 8 belong to the involution determined on a chord of the cubic through the points 1, 2, 3, 4, 5, 6, every line of the system of eight points (Def. 3) is a chord of the cubic determined by the six remaining points; and the points of the system which lie on every such line belong to the involution determined on it by the opposite planes with respect to those six remaining points. And if we take two lines of the system of eight points, the anharmonic ratio of the four planes passing through one of the lines and *the four* remaining points is equal to the anharmonic ratio of the *four planes* passing through the other line and those four remaining

points. The first part of the theorem may be stated more simply by means of symbols, thus:

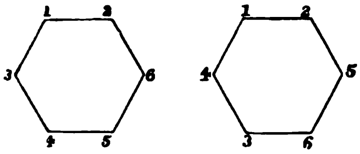
If we express the fact, that 7, 8 belong to the involution determined on a chord of the cubic through 1, 2, 3, 4, 5, 6 by the opposite planes with respect to those points, by the symbolic equation

$$(123456) = (78),$$

then the numbers may be transposed in all possible ways; so that we may write, for example,

$$(712456) = (38).$$

First, any side 12 of any one of the hexagons determined by 1, 2, 3, 4, 5, 6, is a chord of the cubic (345678), drawn through 7, 8 and the remaining points of the hexagon. For 7, 8, the points where the line 78 cuts 126, 345, and the points where 78 cuts 312, 456, are three pairs of points in involution. Therefore, by Lemma C, Cor. 4,



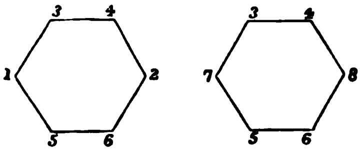
$$12\,(3678) = 45\,(3678).$$

Similarly,
$$12\,(4578) = 36\,(4578).$$

We can, consequently, describe a hyperboloid having 12, 45 as generators, and passing through 3, 6, 7, 8; and another hyperboloid having 12, 36 as generators, and passing through 4, 5, 7, 8. These two hyperboloids have a common generator 12, and the remainder of their intersection is the cubic (345678); therefore by Lemma E, 12 s a chord of this cubic.

Secondly, the points 1, 2 belong to the involution determined on the line 12 by the opposite planes with respect to 3, 4, 5, 6, 7, 8.

For 7, 8, the points where the line 78 cuts 342, 156, and the points where 78 cuts 134, 256, are three pairs of points in involution. Therefore, by Lemma C, Cor. 4,



$$34\,(1278) = 56\,(1278),$$

which may also be written

$$34\,(7812) = 56\,(7812).$$

Therefore, by Lemma C, Cor. 3, the points 1, 2 belong to the

tion determined on the line 12 by the two pairs of planes 348, 567, and 347, 568, *i.e.*, to the involution determined on 12 by the opposite planes with respect to the six points 3, 4, 5, 6, 7, 8.

Thus, from the symbolic equation

$$(123456) = (78),$$

we deduce

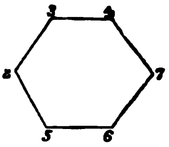
$$(345678) = (12),$$

where the numbers which appear on the right-hand side of the second equation are both taken from among the six numbers which appear on the left-hand side of the first equation. Similarly, from the second equation, we deduce

$$(124568) = (37).$$

Here one number on the right-hand side (7) originally stood on the same side in the first equation; while the other number (3) originally stood on the left-hand side.

Hence it is clear that we may transpose the numbers in all possible ways. This proves the first part of the theorem. To prove the second part: Take any two lines of the system, 34 and 56. Then, by what has been just shown, 1, 8 belong to the involution determined on the line 18 by the two pairs of planes 347, 256 and 234, 567. Therefore, by Lemma C, Cor. 4,

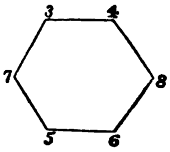


$$34\,(2718) = 56\,(2718).$$

Similarly, the theorem can be proved for any other two lines of the system.

17. *Theorem III.*—If 8 be the eighth homologous point (Def. 1) with respect to the three hyperboloids (12, 34)(5, 6, 7), (34, 56)(1, 2, 7), (56, 12)(3, 4, 7) (see Lemma F), then 78 is a chord of the cubic (123456), and the points 7, 8 belong to the involution determined on the line 78 by the opposite planes with respect to 1, 2, 3, 4, 5, 6.

For, by supposition, 12 is a generator common to two of the hyperboloids. Therefore, by Lemma E, 12 is a chord of the cubic which forms part of the intersection of those two hyperboloids. Now, both hyperboloids pass through 3, 4, 5, 6, 7, 8; therefore the cubic passes through those points; and thus 12 *is a chord* of the cubic (345678).



Again, by the anharmonic property of hyperboloids,

$$34\,(7812) = 56\,(7812);$$

therefore, by Lemma C, Cor. 3, the points 1, 2 belong to the involution determined on the line 12 by the pairs of planes 348, 756 and 347, 568, *i.e.*, by the opposite planes with respect to the system of points 3, 4, 5, 6, 7, 8 (Theorem I.).

Thus we have the symbolic equation

$$(345678) = (12) \quad \text{(see No. 16)};$$

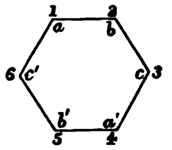
and therefore $$(123456) = (78),$$

or the line 78 is a chord of the cubic (123456), and the points 7, 8 belong to the involution determined on the line 78 by the opposite planes with respect to 1, 2, 3, 4, 5, 6.

Cor.—If 7 lie on one of the planes determined by three of the points 1, 2, 3, 4, 5, 6, then 8 lies on the opposite plane.

18. *Theorem IV.*—If a transversal cut the opposite faces of an octahedron in four pairs of points belonging to one involution, then the transversal is a chord of the cubic determined by the six corners of the octahedron.

This is the converse of Theorem I., and is the second property of gauche cubics which we undertook to prove (see No. 1).



Let the transversal meet the planes 612, 345; 123, 456; 234, 561; 135, 246 (see figure No. 3), in the points a, a'; b, b'; c, c'; d, d', respectively. Then, because dd', bb', cc' are in involution, we have, by Lemma C, Cor. 4,

$$23\,(14dd') = 56\,(14dd').$$

Similarly, because dd', cc', aa' are in involution,

$$34\,(25dd') = 61\,(25dd');$$

and, because dd', aa', bb' are in involution, we have

$$45\,(36dd') = 12\,(36dd').$$

Hence, using the notation given at the end of No. 8, the three hyperboloids (23, 56)(1, 4, d), (34, 61)(2, 5, d), (45, 12)(3, 6, d), which all pass through the seven points 1, 2, 3, 4, 5, 6, and d, and have a common intersection (Lemma D, Cor. 2), all pass through d'. Now this common intersection is a quartic curve, which is, therefore

the plane 246 in four, and only four, points. The four points are, clearly, 2, 4, 6, and d', and there cannot be any other point in the plane 246 common to the three hyperboloids.

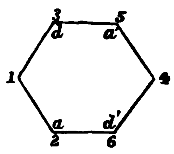
Again, if 7 be any point in space, and if we describe the three hyperboloids (12, 34)(5, 6, 7), (34, 56)(1, 2, 7), (56, 12)(3, 4, 7) as in Lemma F, and determine 8, the eighth homologous point with respect to these three hyperboloids; then, by Theorem III., 78 is a chord of the cubic (123456), and the points 7, 8 belong to the involution determined on the line 78 by the pairs of opposite planes with respect to 1, 2, 3, 4, 5, 6. Consequently, by Theorem II., we have

$$23\,(1478) = 56\,(1478),\quad 34\,(2578) = 61\,(2578),\quad 45\,(3678) = 12\,(3678).$$

This is still true if 7 lie in the plane 135. It follows, if 7 coincide with d, that the point 8 lies on the three hyperboloids (23, 56)(1, 4, d), (34, 61)(2, 5, d), (45, 12)(3, 6, d). But in that case, by the Corollary to Theorem III., 8 lies in the plane 246, and must, by what was shown above, be one of the four points 2, 4, 6, or d'. Hence, as 8 is, in general, distinct from 2, 4, or 6, it will coincide with d'.

It follows, by Theorem III., that dd' is a chord of the cubic (123456).*

* [*6th October*, 1893. If we assume the proposition stated at the beginning of this paper (No. 1), we can give another proof of Theorem IV. Using the notation of No. 18, we have the involution aa', bb', cc', dd'. Let 7 be any point on the transversal and 8 its conjugate in this involution. Then from the involution 78, dd', aa', we deduce



$$35\,(1478) = 26\,(1478).$$

We get, similarly, $13\,(2578) = 64\,(2578)$,

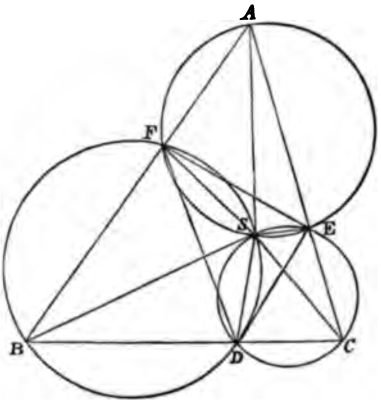
and $51\,(3678) = 42\,(3678)$.

Hence the three hyperboloids (35, 26)(1, 4, 7), (13, 64)(2, 5, 7), and (51, 42)(3, 6, 7), which pass through 1, 2, 3, 4, 5, 6, 7, will all pass through 8. These surfaces, in general, do not pass through a common curve of intersection; therefore they have no other points common, since three quadrics have, in general, eight points common, and no more. But, by the proposition assumed, a system of quadrics passing through seven points will pass also through an eighth point. Therefore the three hyperboloids (12, 34) (5, 6, 7), (34, 56)(1, 2, 7), (56, 12)(3, 4, 7), which pass through 1, 2, 3, 4, 5, 6, 7, pass also through 8. Therefore, by Def. 1, and Lemma F, 8 is the eighth homologous point with respect to the three last hyperboloids. It follows, by Theorem III., that the line 78 is a chord of the cubic through 1, 2, 3, 4, 5, 6.]

Note on the Centres of Similitude of a Triangle of Constant Form Circumscribed to a Given Triangle. By JOHN GRIFFITHS, M.A.

Received June 7th, 1893. Read June 8th, 1893.

If DEF denote a triangle of given species having its vertices D, E, F respectively on the sides BC, CA, AB of a given triangle ABC, or on these sides produced, it is known that DEF will belong to one or other of a pair of systems of similar in-triangles, and that each of



these systems will have a common centre of similitude of its own. (See a "Note on Secondary Tucker-Circles," by the present writer, *Proc. Lond. Math. Soc.*, Vol. XXIV., Nos. 457, 458.)

Now, if we suppose DEF to be *fixed*, and the angles of the circumscribed triangle ABC to be given, the question arises: What is the centre of similitude of ABC?

It is easily seen, by referring to the figure, that if we suppose DEF to be fixed, and the angles of the circum-triangle to be given, the circles CDE, AEF, BFD will be fixed, and their common point of intersection S will be a fixed point.

It is also seen without difficulty that the ratios $SA : SB : SC$ will be fixed. For we have from the triangles SBC, SCA, SAB the following equivalents for the ratios in question, viz.,

$$\frac{SB}{\sin SCB} = \frac{SC}{\sin SBC}, \quad \frac{SC}{\sin SAC} = \frac{SA}{\sin SCA}, \quad \frac{SA}{\sin SBA} = \frac{SB}{\sin SAB}.$$

or

$$\frac{SB}{\sin SED}=\frac{SC}{\sin SFD},\quad \frac{SC}{\sin SFE}=\frac{SA}{\sin SDE},\quad \frac{SA}{\sin SDF}=\frac{SB}{\sin SEF};$$

where the several angles SED, SFD, &c. are given, since DEF is fixed and S is a fixed point.

Again, from the figure, wherein S is supposed to fall *inside* DEF, it appears at once that the angles subtended at S by the sides EF, FD, DE are $\pi-A$, $\pi-B$, $\pi-C$, respectively, and consequently the isogonal coordinates of S with reference to the fixed triangle DEF are

$$x=\frac{\sin A}{\sin(A+D)},\quad y=\frac{\sin B}{\sin(B+E)},\quad z=\frac{\sin C}{\sin(C+F)}.$$

In a similar manner, when S falls *outside* DEF, then the isogonal coordinates of S with reference to DEF are

$$x=\frac{\sin A}{\sin(A-D)},\quad y=\frac{\sin B}{\sin(B-E)},\quad z=\frac{\sin C}{\sin(C-F)}.$$

These results follow at once from the theorem given on p. 130 of the note quoted *supra*, and may be compared with the corresponding expressions for the isogonal coordinates of S with reference to ABC, viz.,

$$x=\frac{\sin(D+A)}{\sin D},\quad y=\frac{\sin(E+B)}{\sin E},\quad z=\frac{\sin(F+C)}{\sin F}.$$

The relation of S to either of the triangles DEF, ABC may thus be determined for given equations connecting the angles of the two triangles. For example, let $D=B$, $E=C$, and $F=A$; then the coordinates of S relatively to DEF, ABC are, respectively, when S falls inside DEF,

$$x=\frac{\sin F}{\sin(F+D)}=\frac{\sin F}{\sin E},\quad y=\frac{\sin D}{\sin F},\quad z=\frac{\sin E}{\sin D},$$

and

$$x=\frac{\sin C}{\sin B},\quad y=\frac{\sin A}{\sin C},\quad z=\frac{\sin B}{\sin A};$$

or S is the centre of similitude of the two triangles, being in fact their common positive Brocard point.

1. It appears from the foregoing propositions that a triangle ABC of given species circumscribed to a fixed triangle DEF belongs to one

or other of two systems of similar circum-triangles, whose centres of similitude are respectively

$$x^{-1} = \frac{\sin(A+D)}{\sin A}, \quad y^{-1} = \frac{\sin(B+E)}{\sin B}, \quad z^{-1} = \frac{\sin(C+F)}{\sin C},$$

and $$x^{-1} = \frac{\sin(A-D)}{\sin A}, \text{ \&c.,}$$

where DEF is supposed to be the triangle of reference.

2. The isogonal, with regard to DEF, of the centre of similitude of a triangle ABC circumscribing DEF, and having a constant Brocard angle, lies on one or other of two given circles whose equations are

$$x \operatorname{cosec} D + y \operatorname{cosec} E + z \operatorname{cosec} F = \cot \omega' \pm \cot \omega,$$

if $$\Sigma \cot D = \cot \omega' \quad \text{and} \quad \Sigma \cot A = \cot \omega.$$

These circles are inverse to each other with respect to the circum-circle of the triangle of reference DEF.

The results in question follow immediately from the fact that if (x, y, z) be the centre of similitude of the circum-triangle ABC, then

$$\Sigma x^{-1} \operatorname{cosec} D = \Sigma (\cot D \pm \cot A),$$

and that the equation of either circle can be transformed into that of the other by writing therein $2\cos D - x$, $2\cos E - y$, $2\cos F - z$ for x, y, z.

The reader will see from the above that theorems similar to those recently obtained by me for the centres of similitude of an in-triangle of constant form will be also true for a circum-triangle of constant form, if for the centre of similitude in the first case we substitute the isogonal point with regard to DEF in the second case. (See *Proc. Lond. Math. Soc.*, Vol. XXIV.)

3. Some additional results with regard to a circum-triangle ABC, of constant form, may be given here, viz. :—

(i.)* As ABC moves, the centre of the circumcircle ABC describes a circle.

This is, in fact, true for any point permanently connected with ABC.

* An exceptional case may be noticed, viz., when $A = D$, $B = E$, $C = F$. T[illegible] centre of the circumcircle ABC then coincides with the orthocentre of the [illegible] triangle DEF in any position of the moveable triangle ABC.

(ii.) The envelope of the circumcircle ABC is a Limaçon having S for a double point.

Otherwise, the circumcircle ABC cuts a given circle J orthogonally, and its centre moves on the circumference of another given circle touching J.

4. The envelope of the radical axis of the circumcircles DEF and ABC is a conic of which the centre of the orthogonal circle J is a focus. In other words, this focus of the conic is also a focus of the Limaçon.

The conic meets the Limaçon in eight points which lie on the circle DEF and another given circle.

On the Harmonics of a Ring. *By* W. D. Niven. Read February 9th, 1893. Received, in revised form, December 11th, 1893.

1. This paper may be regarded as an extension of the work contained in the first memoir on "Toroidal Functions" in the *Philosophical Transactions*, by Professor W. M. Hicks, an attempt being here made to show how the problems solved by Mr. Hicks for a single anchor ring may be dealt with when there are two rings having the same rectilineal axes.

It is obvious that with two such rings, in order to satisfy the surface conditions for each ring, it is necessary that the harmonics of either should be capable of being expressed in terms of those of the other. This is accomplished as follows:—

It is first shown that the harmonics of any degree, referred to a circle A in dipolar coordinates, may be derived from their predecessors, one degree lower, by certain differentiations with regard to the radius of the circle and the distance of its plane from a fixed point.

The potential at any point due to a coaxal circle B of uniform line density is next found in terms of A's harmonics.

Finally, the zonal harmonics of B can all be deduced from this

potential by the system of differentiations just referred to, and therefore they can all be expressed in terms of the harmonics of A.

The present communication is confined to a discussion of the quantities arising in the process indicated, and deals, for the most part, only with zonal harmonics, although the methods employed will also apply to the general tesseral and sectorial system.

2. I begin with the potential at any point due to a circle loaded with matter of line density $\cos\sigma\phi'$ at the point whose longitude along the circumference is ϕ'. If the circle, of radius a, be in the plane of xy, with its centre at the origin of coordinates, the potential at x, y, z or $\rho\cos\phi, \rho\sin\phi, z$ is seen at once to be

$$\sqrt{\frac{a}{2\rho}}\int_0^{2\pi}\frac{\cos\sigma\phi'\,d\phi'}{\sqrt{\dfrac{z^2+\rho^2+a^2}{2a\rho}-\cos(\phi'-\phi)}};$$

or, putting $\phi'-\phi=\theta$, adopting dipolar coordinates u, v, and discarding the term with $\sin\sigma\phi$ as a factor, as being zero, we have

$$\cos\sigma\phi\sqrt{\frac{a}{2\rho}}\int_0^{2\pi}\frac{\cos\sigma\theta}{\sqrt{\coth u-\cos\theta}}\,d\theta,$$

which is the same as

$$\cos\sigma\phi\sqrt{2(\cosh u-\cos v)}\int_0^{\pi}\frac{\cos\sigma\theta}{\sqrt{\cosh u-\sinh u\cos\theta}}\,d\theta.$$

3. Let now the origin be shifted to a point on the rectilineal axis at a distance b from its old position; then the relations between the rectangular and dipolar coordinates are given by

$$\frac{\rho\pm i(z-b)+a}{\rho\pm i(z-b)-a}=e^{u\mp iv},$$

or, if $$a+ib=p,\quad a-ib=q,$$

$$\frac{\rho+iz+q}{\rho+iz-p}=e^{u-iv},\quad \frac{\rho-iz+p}{\rho-iz-q}=e^{u+iv}.$$

If we suppose ρ, z to be fixed, while a, b are varied,[1]

u, v must be regarded as functions of p, q, and it is easy to find the following results:—

$$\frac{\partial u}{\partial p} = \frac{1}{2a} e^{-iv} \sinh u,$$

$$\frac{\partial v}{\partial p} = \frac{1}{2a} i \left(e^{-iv} \cosh u - 1\right),$$

$$\frac{\partial u}{\partial q} = \frac{1}{2a} e^{iv} \sinh u,$$

$$\frac{\partial v}{\partial q} = -\frac{1}{2a} i \left(e^{iv} \cosh u - 1\right),$$

and thence

$$\frac{\partial}{\partial p} \sqrt{\cosh u - \cos v} = \frac{1}{4a} \left(e^{-iv} \cosh u + 1\right) \sqrt{\cosh u - \cos v},$$

$$\frac{\partial}{\partial q} \sqrt{\cosh u - \cos v} = \frac{1}{4a} \left(e^{iv} \cosh u + 1\right) \sqrt{\cosh u - \cos v}.$$

Also, if $I(n, \sigma)$ denote the integral,

$$\int_0^\pi \frac{\cos \sigma\theta}{(\cosh u - \sinh u \cos \theta)^{n+\frac{1}{2}}} \, d\theta,$$

$$\frac{\partial}{\partial p} I(n, \sigma) = -\left(n + \tfrac{1}{2}\right) \frac{1}{2a} e^{-iv} \left[\cosh u \, I(n, \sigma) - I(n+1, \sigma)\right].$$

4. Let $X(n, \sigma)$ denote

$$a^{n-\frac{1}{2}} e^{-niv} \sqrt{2(\cosh u - \cos v)} \, I(n, \sigma) \cos \sigma\phi;$$

then, remembering that $2a = p + q$, we shall have

$$\frac{\partial}{\partial p} X(n, \sigma) = AX(n, \sigma) + BX(n+1, \sigma),$$

where $A = \dfrac{1}{2a}\left(n - \tfrac{1}{2}\right) + \dfrac{1}{2a} n \left(e^{-iv} \cosh u - 1\right) + \dfrac{1}{4a}\left(e^{-iv} \cosh u + 1\right)$

$$-\frac{1}{2a}\left(n + \tfrac{1}{2}\right) e^{-iv} \cosh u = 0,$$

and $$B = \left(n + \tfrac{1}{2}\right) \frac{1}{2a^{\frac{3}{2}}}.$$

Similarly it may be shown that if $Y(n, \sigma)$ denote the same thing as $X(n, \sigma)$, except that e^{niv} takes the place of e^{-niv}, we shall find

$$\frac{\partial}{\partial q} Y(n, \sigma) = (n+\tfrac{1}{2}) \frac{1}{2a^2} Y(n+1, \sigma).$$

Next, let
$$X(n, \sigma) = a^{n-\frac{1}{2}} \{U(n, \sigma) - i V(n, \sigma)\},$$
$$Y(n, \sigma) = a^{n-\frac{1}{2}} \{U(n, \sigma) + i V(n, \sigma)\},$$

and note that
$$\frac{\partial}{\partial a} - i \frac{\partial}{\partial b} = 2 \frac{\partial}{\partial p}, \qquad \frac{\partial}{\partial a} + i \frac{\partial}{\partial b} = 2 \frac{\partial}{\partial q};$$

then the two following relations may be deduced, viz.,

$$(n+\tfrac{1}{2}) a^{n+\frac{1}{2}} U(n+1, \sigma) = a^2 \left\{ \frac{\partial}{\partial a} a^{n-\frac{1}{2}} U(n, \sigma) - \frac{\partial}{\partial b} a^{n-\frac{1}{2}} V(n, \sigma) \right\},$$

$$(n+\tfrac{1}{2}) a^{n+\frac{1}{2}} V(n+1, \sigma) = a^2 \left\{ \frac{\partial}{\partial b} a^{n-\frac{1}{2}} U(n, \sigma) + \frac{\partial}{\partial a} a^{n-\frac{1}{2}} V(n, \sigma) \right\}.$$

Here $U(n, \sigma)$ stands for the harmonic

$$\cos nv \sqrt{2(\cosh u - \cos v)}\, I(n, \sigma) \cos \sigma\phi,$$

and $V(n, \sigma)$ for the same thing, with $\sin nv$ instead of $\cos nv$.

When it is necessary to express u, v, ϕ explicitly, the notation $U_n^\sigma(uv\phi)$, $V_n^\sigma(uv\phi)$ will also be used.

5. These results show that the successive harmonics are derivable from their predecessors by displacements similar, with some modifications, to those usually employed in the case of the sphere. There is this difference, however, that, whereas the masses of the compound sphere are each zero, those of the compound circles, and therefore also of the distributions on the anchor ring which they represent, may not be zero. They will be zero if the expressions for the corresponding potentials contain the longitudinal factor $\cos \sigma\phi$ or $\sin \sigma\phi$, for in that case the undisplaced circle from which they all take their origin is of zero mass. If, however, $\sigma = 0$, the potential $(n+\frac{1}{2}) a^{n-\frac{1}{2}} U(n+1, 0)$ arises, as we found in §4, from displacements of two circles which produce respectively the potentials $a^{n-\frac{1}{2}} U(n, 0)$ and $a^{n-\frac{1}{2}} V(n, 0)$, the former by an enlargement of its radius, the latter by a movement of its centre along its axis. The second displacement is clearly of the same character as sphere points, and no resulting mass is thereby obtained. But, in the first, since the

produces potential $a^{n-\frac{1}{2}}U(n, 0)$ is clearly increased to $\left(1+\frac{\delta a}{a}\right)^{n+\frac{1}{2}}$ times the former value when its radius is increased from a to $a+\delta a$, it follows that by the superposition of ring negative a and positive $a+\delta a$, there will be a residue of mass $(n+\frac{1}{2})\frac{\delta a}{a}$ times the mass of the circle a, viz., $2\pi a \,.\, a^{n-\frac{1}{2}}$. This residue corresponds, by the previous article, to potential

$$(n+\tfrac{1}{2})\, a^{n-\frac{1}{2}}\, U(n+1, 0)\,\frac{\delta a}{a}.$$

Hence, the masses of the compound circles which produce the potentials $U(n, 0)$ and $U(n+1, 0)$ are equal to one another, and when we pursue the argument down to the value $n = 0$, are each equal to $2\pi a$, the line density of the original circle being taken as unity.

6. The result just arrived at throws some light upon the convergence which may be expected from a series expressed in terms of harmonics of the anchor ring, and, on account of its importance, as well as for the purpose of introducing the working quantities of the subject, I shall prove it by finding the charge on the anchor ring which will produce the external harmonic $U(n, 0)$. It is first to be observed that the integral which was denoted in § 3 by $I(n, 0)$ is in point of fact $\pi P_{n-\frac{1}{2}}(\cosh u)$, *i.e.* the zonal spherical harmonic of fractional degree $n-\frac{1}{2}$. The printing of the fractional suffix being inconvenient, I shall at this stage follow Mr. Hicks in representing the integral referred to by πP_n. We may therefore write

$$U(n, 0) \quad \text{or} \quad U_n = \pi \cos nv \sqrt{2\,(\cosh u - \cos v)}\; P_n$$

as the type of a zonal harmonic, with a similar specification for V_n, in which $\sin nv$ takes the place of $\cos nv$.

The harmonic U_n is suitable to the exterior of the ring, the corresponding inside form being given by

$$U'_m = \pi \cos nv \sqrt{2\,(\cosh u - \cos v)}\; Q_n \,.\, \left(\frac{P_n}{Q_n}\right)_{u_0},$$

where $$Q_n = P_n \int_{\mu}^{\infty} \frac{d\mu}{(\mu^2-1)\,P_n^2},$$

$$\mu = \cosh u \quad \text{and} \quad \mu_0 = \cosh u_0,$$

u_0 *being* the parameter of the ring.

We find readily the expression for the density, viz.,

$$\sigma_n = -\tfrac{1}{4}\cos nv\sqrt{2(\cosh u_0 - \cos v)}\frac{1}{\sinh u_0}\frac{1}{Q_n(\cosh u_0)}\left(\frac{\partial u}{\partial s}\right)_{u_0}.$$

Now it is easy to show that

$$-\frac{\partial u}{\partial s} = \frac{\partial v}{\partial s'} = \frac{\cosh u_0 - \cos v}{a},$$

∂s, $\partial s'$ being elements of lines normal to the surfaces u, v. Also, element of surface

$$\partial S = \rho\,\delta\phi\,\delta s' = \frac{a^2 \sinh u_0}{(\cosh u_0 - \cos v)^2}\,\delta\phi\,\delta v.$$

Hence $$\iint \sigma_n\,dS = 2\pi a\frac{1}{Q_n^{\cdot}(u_0)}\int_0^{\pi}\frac{\cos nv}{\sqrt{2(\cosh u_0 - \cos v)}}\,dv$$

$$= 2\pi a \quad (\S\,14).$$

7. The potential due to a circle of unit line-density with its centre on the rectilineal axis of the ring and its plane normal to that axis, may be found as follows:—Let r be the distance of a point $u_1v_1\phi_1$ on the circle referred to from any point $uv\phi$ on the ring; then

$$\frac{1}{r} = \Sigma\Sigma A_n^{\sigma} U_n^{\sigma}(uv\phi) + \Sigma\Sigma B_n^{\sigma} V_n^{\sigma}(uv\phi).$$

The expression for the circle, since u_1, v_1 are the same for every point of it, may be deduced from this by multiplying by $2\pi\rho$, and suppressing all the terms in $\cos\sigma\phi$ and $\sin\sigma\phi$ from $\sigma = 1$ upwards.

The coefficients to be determined in the expansion of $\frac{1}{r}$ are therefore only the following

$$A_0U_0 + A_1U_1 + A_1'V_1 + \dots .$$

Multiply both sides by $\sigma_n dS$, and integrate over the surface, observing that

$$\iint \frac{\sigma_n dS}{r} = U_n(u_1v_1),$$

and that, by § 6,

$$\iint \sigma_n U_n dS = \pi^2 a\left(\frac{P_n}{Q_n}\right)_u \int_0^{2\pi}\cos^2 nv\,dv.$$

Hence $$A_n = \frac{1}{\pi^2 a}U_n(u_1v_1)\left(\frac{Q_n}{P_n}\right)_u,$$

except when $n = 0$, and then

$$A_0 = \frac{1}{2\pi^2 a} U_0(u_1 v_1) \left(\frac{Q_0}{P_0}\right)_n.$$

If Q_n be substituted for P_n in the expressions for U_n and V_n, the resulting harmonic is finite in the neighbourhood of the dipolar circle of reference. We shall denote these forms of the harmonic by $\overline{U}_n$ and $\overline{V}_n$ respectively. The expansion for the potential at $uv\phi$ due to the circle of the last article will then be written as follows :—

$$\frac{\rho}{\pi^2 a} \left\{ U_0(u_1 v_1)\, \overline{U}_0(uv) + 2\sum_1^\infty \left[U_n(u_1 v_1)\, \overline{U}(uv) + V_n(u_1 v_1)\, \overline{V}_n(uv) \right] \right\}.$$

Problems with Two Rings.

8. The last result, taken in conjunction with the results of § 4, enables us to express the harmonics of one anchor ring in terms of those of another which is coaxal with it. For it was shown in § 4 that all the harmonics of an anchor ring could be deduced from the potential of a circle by certain rules of differentiation. We have therefore only to apply those rules to the expansion just found in order to obtain similar expansions, in terms of the harmonics of the circle a, for the series of compound circles of radius ρ, that is to say, for the complete system of harmonics pertaining to the system of anchor rings the radius of whose dipolar circle is ρ.

In order to produce a symmetrical arrangement, we may suppose the positive directions of the rectilineal axes of the two rings to be towards one another, so that, if A and B be the two rings, the displacements of A are towards B, and those of B towards A. When this arrangement is turned round, so that A and B interchange planes, the relative arrangement of harmonics will then be the same as before.

9. Let W_0, $W_1 \mathcal{W}_1$, $W_2 \mathcal{W}_2$, &c. represent the harmonics of B, referred to its dipolar circle and expressed in terms of its own dipolar system, then, § 9,

$$W_0 = \frac{\rho}{\pi^2 a} \left\{ U_0(u_1 v_1)\, \overline{U}_0(uv) + 2U_1(u_1 v_1)\, \overline{U}_1(uv) + \&c. \right\}$$

$$= \frac{1}{\pi^2} \left({}_0\alpha_0\, \overline{U}_0 + {}_0\alpha_1\, \overline{U}_1 + {}_0\beta_1 \overline{V}_1 + \ldots \right), \text{ say,}$$

where
$$ {}_0\alpha_0 = \frac{\rho}{a}\, U_0\,(u_1v_1), $$
$$ {}_0\alpha_1 = \frac{2\rho}{a}\, U_1\,(u_1v_1), $$
$$ {}_0\beta_1 = \frac{2\rho}{a}\, V_1\,(u_1v_1), $$
&c.,

and $\overline{U}_n$ is put, for brevity, instead of $\overline{U}_n\,(uv)$, &c. All the coefficients which have zero for the first suffix can be readily expressed in terms of the radii ρ and a and the distance z between the planes of the circles. It may, in fact, be shown that
$$ U_n\,(u_1v_1)+iV_n\,(u_1v_1) = 2a\,(\rho^2+z^2-a^2+i\,2az)^n\,J_n, $$
where
$$ J_n = \int_0^\pi \frac{d\theta}{(\rho^2+z^2+a^2-2a\rho\cos\theta)^{n+\frac{1}{2}}}. $$

For, since
$$ e^{u\mp iv} = \frac{\rho\pm iz+a}{\rho\pm iz-a}, $$
we have
$$ e^u = r_1/r, $$
where
$$ r^2 = (\rho-a)^2+z^2, \quad r_1^2 = (\rho+a)^2+z^2, $$
and
$$ e^{2iv} = \frac{\rho^2+z^2-a^2+i\,2az}{\rho^2+z^2-a^2-i\,2az}; $$
therefore
$$ \sinh u = \frac{2a\rho}{rr_1}, $$
and
$$ \cos nv = \tfrac{1}{2}\,\frac{(\rho^2+z^2-a^2+i\,2az)^n+(\rho^2+z^2-a^2-i\,2az)^n}{(rr_1)^n}. $$

Hence
$$ U^n\,(u_1v_1) = \frac{\cos nv}{\rho}\,\frac{(2a\rho)^{n+1}}{\sinh^n u}\int_0^\pi \frac{d\theta}{(\rho^2+z^2+a^2-2a\rho\cos\theta)^{n+\frac{1}{2}}} $$
$$ = a\left\{(\rho^2+z^2-a^2+i\,2az)^n+(\rho^2+z^2-a^2-i\,2az)^n\right\}J_n\,; $$
similarly,
$$ V_n\,(u_1v_1) = ia\left\{(\rho^2+z^2-a^2-i\,2az)^n-(\rho^2+z^2-a^2+i\,2az)^n\right\}J_n. $$

10. The next step is to determine the expansions of W_1 and $\mathfrak{W}_1$. If we put
$$ \pi^2 W_1 = {}_1\alpha_0\,\overline{U}_0+{}_1\alpha_1\,\overline{U}_1+{}_1\beta_1\,\overline{V}_1+\text{\&c.}, $$
$$ \pi^2 \mathfrak{W}_1 = {}_1\gamma_0\,\overline{U}_0+{}_1\gamma_1\,\overline{U}_1+{}_1\delta_1\,\overline{V}_1+\text{\&c.}, $$

then, by § 4 and § 9, ${}_1\alpha_0$, ${}_1\alpha_1$, ${}_1\beta_1$ will be respectively

$$\frac{1}{a}\left(2\rho\frac{\partial}{\partial\rho}-1\right)\rho\left\{U_0(u_1v_1),\ 2U_1(u_1v_1),\ 2V_1(u_1v_1),\ \ldots\right\};$$

and ${}_1\gamma_0$, ${}_1\gamma_1$, ${}_1\delta_1$, &c. respectively

$$-\frac{2\rho^2}{a}\frac{\partial}{\partial z}\left\{U_0(u_1v_1),\ 2U_1(u_1v_1),\ 2V_1(u_1v_1),\ \ldots\right\}.$$

11. To show, however, how the coefficients are formed in the general case, it should be noticed that the equations of § 4 may be thrown into the form

$$(2n+1)(U_{n+1}+iV_{n+1})=\left(2a\frac{\partial}{\partial a}+2n-1+i\,2a\frac{\partial}{\partial b}\right)(U_n+iV_n).$$

If the harmonics be those at the point ρ, z, ϕ, referred to the circle A, it is clear that we may write $-\frac{\partial}{\partial z}$ for $\frac{\partial}{\partial b}$ in this relation. Next, putting Ω for the operator $2a\frac{\partial}{\partial a}-i\,2a\frac{\partial}{\partial z}$, and Ω' for the same operator with the sign of i changed, we shall have

$$U_n+iV_n=\frac{(\Omega+2n-3)(\Omega+2n-5)\ldots(\Omega-1)}{(2n-1)(2n-3)\ldots 1}U_0$$

$$=f_n(\Omega)\,2aJ_0,\ \text{suppose.}$$

Next, as regards the harmonics W, as the displacements of B are towards A, the sign of $\frac{\partial}{\partial b}$ in § 4 must be changed, and, since b may then be put equal to z, we have in like manner

$$W_n+i\mathcal{W}_n=f_n(\varpi)\,W_0,$$

where ϖ is put for $2\rho\frac{\partial}{\partial\rho}-i\,2\rho\frac{\partial}{\partial z}$, and ϖ' may be taken to represent the same operator with the sign of i changed.

If now
$$W_n=\Sigma\,{}_n\alpha_r\,\overline{U}_r+\Sigma\,{}_n\beta_r\,\overline{V}_r,$$
$$\mathcal{W}_n=\Sigma\,{}_n\gamma_r\,\overline{U}_r+\Sigma\,{}_n\delta_r\,\overline{V}_r,$$
we shall have

$${}_n\alpha_r+i\,{}_n\gamma_r=f_n(\varpi)\frac{2\rho}{a}U_r(u_1v_1),$$

$${}_n\beta_r+i\,{}_n\delta_r=f_n(\varpi)\frac{2\rho}{a}V_r(u_1v_1),$$

except when $r=0$, when the 2 on the right-hand side must be omitted.

12. The equations just written down determine the coefficients completely. As a matter of convenience, however, it may happen that it is easier to find, for instance, ${}_na_r$ from ${}_na_0$ than from ${}_0a_r$.

The coefficient ${}_na_0$ is itself to be found from

$$\tfrac{1}{2}\left\{f_n(\varpi)+f_n(\varpi')\right\}\frac{\rho}{a}U_0,$$

i.e.

$$\left\{f_n(\varpi)+f_n(\varpi')\right\}\rho J_0.$$

Now J_0 is a symmetrical function of ρ and a, and the operation to be performed is precisely the same as that for finding ${}_0a_n$, except that ρ and a are everywhere interchanged. Hence, if for a minute we denote the expression for ${}_0a_n$, found above, by $F(a, \rho, z)$, that for ${}_na_0$ will be $F(\rho, a, z)$. Now

$$\begin{aligned}{}_na_r &= \left\{f_n(\varpi)+f_n(\varpi')\right\}\frac{\rho}{a}U_r(u_1v_1)\\ &= \left\{f_n(\varpi)+f_n(\varpi')\right\}\frac{\rho}{a}\left\{f_r(\Omega)+f_r(\Omega')\right\}aJ_0\\ &= \frac{1}{a}\left\{f_r(\Omega)+f_r(\Omega')\right\}a\left\{f_n(\varpi)+f_n(\varpi')\right\}\rho J_0\\ &= \frac{1}{a}\left\{f_r(\Omega)+f_r(\Omega')\right\}a\,{}_na_0.\end{aligned}$$

13. The integral J_n, which constantly appears in the coefficients, is only a modification of P_n, and both P_n and Q_n, or their first differential coefficients, occur in the equations expressing the boundary conditions in any physical problem, connected with the ring, to which potential functions are applicable. It is therefore important to be able to evaluate these integrals. Mr. Hicks has shown how to express them in terms of elliptic integrals; in what follows it is shown how they may be expressed in series, a form which seems to possess advantages when the section of the ring is small.

Evaluation of the Integrals P_n, Q_n.

The integral $I(n, \sigma)$, which appeared in § 3, is one factor of tesseral harmonic, but it will be sufficient at present to confine o

attention to zonal forms. We had

$$\pi P_n = \int_0^{\pi} \frac{d\theta}{(\cosh u - \sinh u \cos\theta)^{n+\frac{1}{2}}} \quad\text{..................(1),}$$

which is readily thrown into the form

$$2e^{-(n+\frac{1}{2})u} \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\left\{1-(1-e^{-2u})\sin^2\theta\right\}^{n+\frac{1}{2}}} \quad\text{..............(2).}$$

14. The corresponding form for Q_n will now be found. It is to be observed that we have already defined Q_n, § 6, but another form can be found for it from the external zonal harmonic of a prolate ellipsoid, and we shall, in fact, show afterwards that the two are connected by the relation

$$Q_n = P_n \int_{\mu}^{\infty} \frac{d\mu}{(\mu^2-1)\,P_n^2} = \frac{1}{2^{n+\frac{1}{2}}} \int_{-1}^{1} \frac{(1-v^2)^{n-\frac{1}{2}}}{(\mu-v)^{n+\frac{1}{2}}}\,dv \quad\text{.........(3),}$$

where $\mu = \cosh u$.

From the second of these we at once pass to

$$\frac{(-1)^n}{1.3.5\ldots(2n-1)} \int_{-1}^{1} \frac{\dfrac{\partial^n}{\partial v^n}(1-v^2)^{n-\frac{1}{2}}\,dv}{\sqrt{2}\,(\mu-v)},$$

and then to

$$\int_0^{\pi} \frac{\cos n\theta}{\sqrt{2}\,(\cosh u - \cos\theta)}\,d\theta \quad\text{......................(4).}$$

From (3) we also get

$$2^{-n-\frac{1}{2}} \int_0^{\pi} \frac{\sin^{2n}\theta}{(\cosh u - \cos\theta)^{n+\frac{1}{2}}}\,d\theta \quad\text{....................(5),}$$

and a still more useful expression for Q_n may be obtained by employing a transformation similar to Landen's, viz.,

$$\sin(\psi-\theta) = e^{-u}\sin\psi.$$

We thus find, on reduction,

$$Q_n = e^{-(n+\frac{1}{2})u} \int_0^{\pi} \frac{\sin^{2n}\psi}{\sqrt{1-e^{-2u}\sin^2\psi}}\,d\psi \quad\text{.................(6).}$$

This corresponds to the expression (2), § 13, for P_n, and furnishes a convenient expansion in powers of e^{-u}.

*15. I shall now find the expansion of P_n in powers of e^u and e^{-u}.

* Since this was written, Mr. A. B. Basset, in a paper on "Toroidal Functions," in the *American Journal of Mathematics*, has given the expansion of P_n. The work in the text is, however, retained, as the method is somewhat different.

If y be put for Q_n/π, which may be written

$$e^{-\frac{1}{2}u}(a_n e^{-nu}+a_{n+2}e^{-(n+2)u}+\ldots),$$

where a_n, a_{n+2}, &c. are known coefficients, then y satisfies the differential equation

$$\frac{d^2y}{du^2}+\coth u\,\frac{dy}{du}-(n^2-\tfrac{1}{4})\,y=0,$$

and, after Cayley (*Elliptic Functions*, § 77), there exists another solution of the form

$$(\log 4+u)\,y+z,$$

where z satisfies the differential equation

$$(e^u-e^{-u})\frac{d^2z}{du^2}+(e^u+e^{-u})\frac{dz}{du}-(n^2-\tfrac{1}{4})(e^u-e^{-u})\,z$$

$$=-2\,(e^u-e^{-u})\frac{dy}{du}-(e^u+e^{-u})\,y \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$=-e^{-\frac{1}{2}u}\,(b_{n-1}e^{-(n-1)u}+b_{n+1}e^{-(n+1)u}+\ldots)\ \text{suppose},$$

where b_{n-1}, b_{n+1}, &c. are known coefficients.

Let $z=e^{-\frac{1}{2}u}\left[A_n e^{nu}+A_{n-2}e^{(n-2)u}+\ldots+A_{n-2r}e^{(n-2r)u}+\ldots\right.$

$$\left.+B_{n-2r}e^{-(n-2r)}+\ldots+B_{n+2s}e^{-(n+2s)u}+\ldots\right).$$

If n be even, there will be a term A_0, or B_0, which means the same thing; if n be odd, there will be a term A_1e^u and a term B_1e^{-u}. The values of r may range from $\frac{n}{2}$ (n even) or $\frac{n-1}{2}$ (n odd) to 0, and s from 1 to ∞.

Entering the value of z in the differential equation, we find

$$A_{n-2r+2}=A_{n-2r}\frac{4r\,(n-r)}{(2r-1)(2n-2r+1)}\quad\ldots\ldots\ldots\ldots\ldots(2),$$

and

$$B_{n-2r-2}=B_{n-2r}\frac{4r\,(n-r)}{(2r+1)(2n-2r-1)}\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

One or other of these equations holds from the top of the series A_n down to B_{n-4} inclusive. After that we begin to take account of the second side of equation (1). It will be found that

$$B_{n-2}=-\frac{b_{n-1}}{2n-1}=\frac{2n}{2n-1}a_n\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4);$$

but after this point we shall have equations of the form

$$(2s-1)(2n+2s-1)\,B_{n+2s-2}-4s\,(n+s)\,B_{n+2s}=b_{n+2s-1}\quad\ldots\ldots(5),$$

which it will be impossible to solve unless we can assume the value of one B of the system to be known. To get over this difficulty, I

suppose B_n temporarily withdrawn, and all the coefficients after it determined by the equations (5). We may then restore B_n in the form of a term $B_n y$ in the expression for z, where y is the solution referred to at the beginning of this article. All the coefficients in z are now found from the equations (5) solved successively, with the exception of B_n, and this may be determined by taking $k^2 = k_1^2 = \frac{1}{2}$, calculating independently the corresponding values of P_n, Q_n for this value of k^2, and entering these values in the equation

$$\pi P_n = (B_n + \log 4 + u)\, Q_n + \pi z,$$

where z does not now include B_n. The term B_n may, however, be easily found in the cases $n = 1$, 2, by differentiation of $F(k)$ with regard to k^2.

When n is even, (2) gives

$$A_n = A_0 2^{2n-1} \frac{n!\,n!}{(2n)!},$$

and
$$B_0 = B_{n-2} \frac{2^{n-1}(n-1)!}{1.3\ldots(2n-3)} = \frac{2^{2n}\,n!\,n!}{n\,(2n)!}\,a_n = \frac{1}{n};$$

therefore
$$A_n = 2^{2n-1} \frac{n!\,(n-1)!}{(2n)!}.$$

The same expression holds when n is odd, the transition from the A's to the B's being effected by the equation

$$n^2 A_1 = (n^2-1)\, B_1,$$

except when $n = 1$.

16. When n is large, the first term is the most important in the expansion; the series will therefore conveniently proceed from that term.

If we put $e^{-u} = k_1$ and $k^2 = 1 - k_1^2$, then, § 14,

$$\pi P_n = 2k_1^{n+\frac{1}{2}}\, F_n(k),$$

$F_n(k)$ being written for

$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1-k^2\sin^2\theta)^{n+\frac{1}{2}}},$$

and
$$F_n(k) = 2^{2n-1} \frac{n!\,(n-1)!}{(2n)!} k_1^{-2n} \left\{ 1 + \frac{1\,(2n-1)}{4.1\,(n-1)} k_1^2 + \frac{1.3\,(2n-1)(2n-3)}{4^2.1.2\,(n-1)(n-2)} k_1^4 + \&c. \right\}$$
$$+ \left(B_n + \log \frac{4}{k_1}\right) \frac{(2n)!}{2^{2n}\,n!\,n!} \left\{ 1 + \frac{2n+1}{n+1} k_1^2 + \ldots \right\},$$

where B_n is to be determined by the method of § 15.

17. J_n, being $$= \frac{1}{r^{2n+1}}\, 2\int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1-k^2\sin^2\theta)^{n+\frac{1}{2}}},$$

where $$k_1^2 = 1-k^2 = r^2/r_1^2,$$

is also found by this expansion. A few terms of the earlier values of J_n, in which B_n appears at an early stage of the series, may be obtained by the method of differentiation previously referred to. The series are arranged in powers of r/r_1, supposed small:—

$$J_0 = \frac{2}{r_1}\log\frac{4r_1}{r} + \frac{1}{2}\,\frac{r^2}{r_1^3}\left(\log\frac{4r_1}{r} - 1\right) + \&c.,$$

$$J_1 = \frac{2}{r_1 r^2} + \frac{1}{r_1^3}\left(\log\frac{4r_1}{r} - \frac{1}{2}\right) + \&c.,$$

$$J_2 = \frac{4}{3r_1 r^4} + \frac{1}{r_1^3 r^2} + \frac{3}{4}\,\frac{1}{r_1^5}\left(\log\frac{4r_1}{r} - \frac{7}{12}\right).$$

18. The expansion for P_n enables us to justify the assumption made in § 14. For let

$$P_n\int_\mu^\infty \frac{d\mu}{(\mu^2-1)\,P_n^2} = A\,e^{-(n+\frac{1}{2})u}\int_0^\pi \frac{\sin^{2n}\psi}{\sqrt{1-e^{-2u}\sin^2\psi}}\,d\psi,$$

where A is a constant to be determined. If we suppose u to be so large that the first term of P_n need only be considered, and also that $2\sinh u$ may be put equal to e^u, then we shall have

$$A = 1,$$

thus proving the equality of the two expressions for Q_n assumed in § 14.

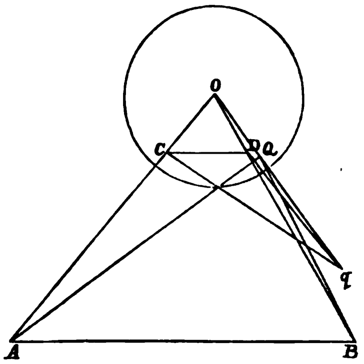
Sphere and Ring.

19. A similar process of solution may be applied when, instead of the ring B, we substitute a sphere with its centre on the rectilineal axis of the ring A.

Referring to the figure, we suppose C, D to be the inverted positions of A, B, the extremities of the diameter of the dipolar axis of the ring. Let the angle subtended by this diameter at the centre O be denoted by 2α, and let Q be any point inside the sphere, q its

inverted position. Then, by similar triangles,

$$\frac{Cq}{Oq} = \frac{AQ}{AO}, \text{ and } \frac{Dq}{Oq} = \frac{BQ}{BO};$$



therefore
$$\frac{Cq}{Dq} = \frac{AQ}{BQ}.$$

Hence, if u', v' be the dipolar coordinates of q referred to CD, while those of Q are u, v referred to AB, we have

$$u' = u.$$

Also, from the geometry of the figure, we readily find that, if q is below CD,

$$v' = v - 2a,$$

and, if above,
$$v' = 2a - v.$$

Moreover, $OQ^2 = \rho^2 + (c \cot a - z)^2 = \dfrac{c^2}{\sin^2 a} \dfrac{\cosh u - \cos(v-2a)}{\cosh u - \cos v}$.

Hence any harmonic, say

$$\pi \cos mv \sqrt{2(\cosh u - \cos v)}\, P_n(\cosh u),$$

becomes, by the rule of inversion, if q is below CD,

$$\pi \cos m(v_1 + 2a) \sqrt{2\{\cosh u_1 - \cos(v_1 + 2a)\}}\, P_n(\cosh u_1)$$

$$\times \frac{c}{R \sin a} \sqrt{\frac{\cosh u_1 - \cos v_1}{\cosh u_1 - \cos(v_1 + 2a)}},$$

viz.,
$$\frac{c}{R \sin a} \cos m(v' + 2a) \sqrt{2(\cosh u_1 - \cos v_1)}\, P_n(\cosh u_1);$$

and, if q is above CD, the same expression with $2a - v_1$ written instead of $v' + 2a$.

This result will greatly simplify the electrostatic problem for a sphere and ring.

The following presents to the Library were received during the recess :—

"Vierteljahrschrift der Naturforschenden Gesellschaft in Zurich," 38[er] Jahrgang, Heft 1 and 2; 1893.

"Proceedings of the Edinburgh Mathematical Society," Vol. XI.; Session 1892–3.

"Journal of the Institute of Actuaries," Vol. XXX., Pt. 6, July, 1893; Vol. XXXI., Pt. 1, October, 1893.

"Proceedings of the Royal Society," Vol. LIII., No. 322–325; Vol. LIV., No. 326.

"Beiblätter zu den Annalen der Physik und Chemie," Band XVII., Stücke 5–8; Leipzig, 1893.

"Nyt Tidsskrift for Mathematik," A. Fjerde Aargang, No. 3; B. Fjerde Aargang, No. 2; Copenhagen, 1893.

"Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig," Mathematisch-Physische Classe, 1893, 2 and 3.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 4[me] Serie, Tomes I. and III., 1[ère] cahier.

"Memoirs and Proceedings of the Manchester Literary and Philosophical Society," Vol. VII., Nos. 2 and 3; 1892–3.

"Nieuw Archief voor Wiskunde," Deel XX., Stuk 2; Amsterdam, 1893.

"Archives Néerlandaises des Sciences Exactes et Naturelles," Tome XXVII., Livraisons 1 and 2; Harlem, 1893.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XI., No. 4; Coimbra, 1893.

"Bulletin de la Société Mathématique de France," Tome XXI., Nos. 4, 5, 6.

"Bulletin of the New York Mathematical Society," Vol. II., Nos. 9 and 10; 1893.

"Bulletin des Sciences Mathématiques," Tome XVII., Avril, Mai, Juin, et Juillet; Paris, 1893.

"Mathematical Questions, with their Solutions," edited by W. J. C. Miller, Vol. LIX.; London, 1893.

"Treatise on the Kinetic Theory of Gases," by H. W. Watson, 2nd edition, 8vo; Oxford, 1893.

"Treatise on the Mathematical Theory of Elasticity," by A. E. H. Love, Vol. II., R. 8vo; Cambridge, 1893.

"Ueber einige Eigenschaften der Bessel'schen Function erster Art, insbesondere für ein grosses Argument," von Dr. J. H. Graf. (Offprint from "Zeitschrift für Mathematik und Physik," aus dem 2 Hefte des 38 Jahrgangs.)

"Ueber die Addition und Subtraction der Argumente bei Bessel'schen Functionen, nebst einer Anwendung," von Dr. J. H. Graf. (Offprint from "Math. Annalen," Vol. XLIII., pp. 136–44.)

"Wiskundige Opgaven met de Oplossingen," Zesde Deel, Stuk 1; Amsterdam, 1893.

"Revue Semestrielle des Publications Mathématiques," Tome I., 2[ème] partie; Amsterdam, 1893.

"Observations Pluviométriques et Thermométriques dans le Département de la Gironde, de Juin, 1891, à Mai, 1892," par Mons. G. Rayet; Bordeaux, 1892.

"Prace Matematyczno-Fizyczne," Tom. IV.; Warsaw, 1893.

"Sphärische Trigonometrie, orthogonale Substitutionen, und elliptische Functionen," von E. Study; Leipzig, 1893.

Mons. M. d'Ocagne.—"Sur la détermination géométrique du point le plus probable donné par un système de droites non convergentes." (Extrait du "Journal de l'Ecole Polytechnique," LXIII.^e cahier, 1893.)

"Jahrbuch über die Fortschritte der Mathematik," Bd. XXII., Heft 3; Berlin, 1893.

"Journal of the Japan College of Science," Vol. V., Part 4; Vol. VI., Part 2; Tokyo, 1893.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," Parts 1–38, 1893.

"Atti della Reale Accademia dei Lincei," 5^a Serie, Rendiconti, Vol. II., Fasc. 1, 2, 4, 5, 6, 8, 9, 10, 11, 12; Roma, 1893.

"Atti della Reale Accademia dei Lincei," Anno CCXC., Rendiconti; Roma, 1893.

"Journal für die reine und angewandte Mathematik," Bd. CXII., Hefte 1–4.

"Annals of Mathematics," Vol. VII., No. 4; May, 1893; University of Virginia.

"Annales de la Faculté des Sciences de Toulouse," Tome VII., Année 1893, 2^me Fasc.

"Annali di Matematica," Tomo XXI., Fasc. 2, 3; Milano, 1893.

"Educational Times," July to October, 1893.

"Indian Engineering," Vol. XIII., Nos. 20–25, and Vol. XIV., Nos. 1–12.

"Rendiconti dell' Accademia delle Scienze Fisiche e Matematiche di Napoli," Serie 2, Vol. VII., Fasc. 5–7; Napoli.

"Atti della Reale Accademia delle Scienze Fisiche e Matematiche di Napoli," Serie 2, Vol. V., 1893.

"Electrical Engineer," No. 24, Vol. XI.; June, 1893.

"Annuaire de l'Académie Royale de Belgique," 1892–3, Bruxelles.

"Bulletin de l'Académie Royale de Belgique," 1891–2, Parts 1, 2, 1893, Bruxelles.

NOTES.

Mr. Basset's paper on "Toroidal Functions" (p. 180) appears, in substance, in the *American Journal of Mathematics*, Vol. xv., No. 4 (pp. 287–302).

Mr. Larmor sends the following corrections :—

In the account of magnetic optical rotation (*Proceedings*, 1893, p. 280, *seq.*) there are some points that require correction. Prof. Willard Gibbs (*loc. cit.*, p. 114) does not appear to contemplate the extension of his equation, between electric flux and force, to express the influence of a magnetic field. The statements on p. 285, as to the final agreement of different theories, are put more correctly in a paper on "The Theory of Magnetic Action on Light, . . ." in the *Report of the British Association* for 1893.

On p. 284, line 17, *for* $\kappa_3 \dfrac{d^2\theta}{dz^2}$, *read* $\kappa_2 \dfrac{d^2\theta}{dt^2}$; and on line 26, *for* "kinetic energy," *read* "force per unit volume."

I am indebted to Mr. Heaviside for pointing out that on p. 278, line 6 from the foot, the index should be $+\frac{1}{2}$ instead of $-\frac{1}{2}$; also that ambiguity would be avoided, on line 2 from the foot, by an explicit statement that the Fresnel surface mentioned has $K_1'\mu_2'\mu_3'$, $K_2'\mu_3'\mu_1'$, $K_3'\mu_1'\mu_2'$ for the squares of its principal velocities.

Dr. P. Zeeman informs me that the result of an extended series of experiments, recently conducted by him, is to show that the circumstances of magnetic reflexion cannot be represented by one optical constant, as in Drude's theory; but that they can be represented very well by aid of two such constants, after the manner of Goldhammer. Dr. Zeeman holds that in his experiments the influence of surface layers could not possibly be so great as to account for the discrepancy.

Errata.

P. 162, line 5 up, *for* *LB* *read* $\angle B$.

Vol. xviii., p. 397, lines 2, 3 up, dexter side of equation should be multiplied by 2.

Vol. xxi., p. 5, line 13 up, *read* $\lambda + \lambda' = 1$.

,, p. 447, line 2 up, *for* R *read* R_q.

INDEX.

INDEX OF PAPERS, &c.

[*The mark * indicates that the Paper has either been printed* in extenso *or an abstract of it given in the "Proceedings."*]

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